A Complete Equational Presentation of Qudit Circuits

via polycontrolled PROPs

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Qudits are one system with d named states

Definition

A qudit wire represents a d-dimensional space with computational basis

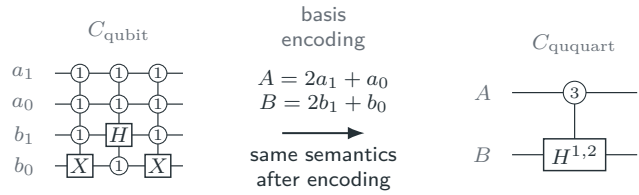
$$|0\rangle, |1\rangle, \ldots, |d-1\rangle.$$

An n-wire circuit represents a unitary on $(\mathbb{C}^d)^{\otimes n}$.

Qudit platforms

- ion levels: trapped-ion qudits ($d \leq 7$) [Ringbauer 22] [Hrmo 23]
- photonic modes: integrated ququart processors [Chi 22]
- transmon levels: qutrit/ququart control [IBM] [Fischer 23]
- bosonic modes: cat-code systems, related but not qudits [Alice & Bob]

Why qudits? More structure per wire



Compression example

Packing two qubits into one ququart turns the branch 11 into the value $A = 3$, and the target pair $01, 10$ into ququart levels $1, 2$.

Circuit-level use cases

wire compression	pack binary data into higher-dimensional wires [Litteken 23]
temporary levels	store intermediate values during Toffoli decompositions [Gokhale 19] [Nikolaeva 23]

One-qudit universality comes from two-level gates

Embedding a 2×2 unitary

$$U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}.$$

$U^{(i,i+1)}$ is the identity except on adjacent rows and columns $i, i+1$:

$$U^{(i,i+1)} = \begin{bmatrix} I & & & \\ & u_{00} & u_{01} & \\ & u_{10} & u_{11} & \\ & & & I \end{bmatrix}$$

rows and columns ordered as $\cdots, i, i+1, \cdots$

One-qudit synthesis

On the active two-level subspace:

$$P_0(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix},$$
$$P_1(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

Euler angles write any $U(2)$ operation as a product of

$$P_0(\alpha),\ P_1(\beta),\ H.$$

Adjacent two-level embeddings generate $U(d)$. [Nielsen–Chuang] [Clements 16]

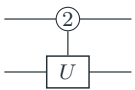
One entangling gate gives n-qudit universality

Entangling control

A value-control is a direct-sum branch.
For $d = 4$,

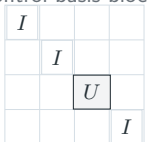
$$\mathrm{ctrl}_2(U) = I \oplus I \oplus U \oplus I.$$

active branch $k = 2$



$\mathrm{ctrl}_2(U)$

control-basis blocks

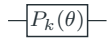


$\mathrm{diag}(I, I, U, I)$

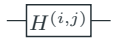
Universal gate set $\mathcal{G}_d$ *[Brylinski–Brylinski 02]*

$$\mathcal{L}_d = \{P_k(\theta),\ H^{(i,j)}\},$$
$$\mathcal{G}_d = \mathcal{L}_d \cup \{\mathrm{ctrl}_1(P_1(\theta))\},$$
$$\langle \mathcal{G}_d \rangle = U(d^n).$$

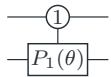
one-qudit



one-qudit



entangling



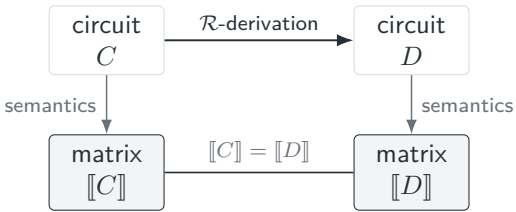
What is a complete presentation?

Definition

A presentation is a gate signature Σ together with equations $\mathcal{R}$, closed under circuit contexts.

$$\mathcal{R} \vdash C = D$$

means that C can be rewritten to D by local rule applications.



Soundness: derivations preserve semantics.
Completeness: equal semantics give a derivation.

Example

Two adjacent H gates cancel; the same local rewrite is allowed inside any circuit context.

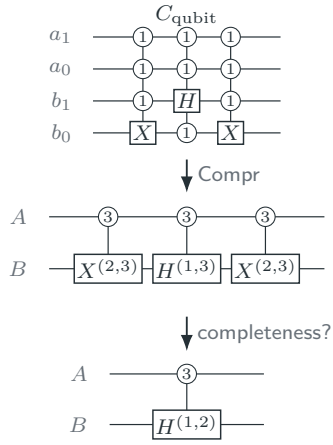
Compressing four basis states into one ququart

Basis encoding

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \overset{\text{enc}}{\cong} \mathbb{C}^4, \quad |a_1 a_0\rangle \mapsto |2a_1 + a_0\rangle$$

Compression of 4 *basis states*

$\mathsf{Compr}(D \circ C) = \mathsf{Compr}(D) \circ \mathsf{Compr}(C)$,
$\mathsf{Compr}(\mathrm{ctrl}_1(\mathrm{ctrl}_1(U))) = \mathrm{ctrl}_3(\mathsf{Compr}(U))$,
$\mathsf{Compr}(\mathrm{ctrl}_1(X)) = X^{(2,3)}$,
$\mathsf{Compr}(\sigma \circ \mathrm{ctrl}_1(H) \circ \sigma) = H^{(1,3)}$.
△ This function here is partially defined.

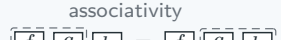


A PRO formalizes circuit composition

Definition

A **PRO** is a strict monoidal category $\mathcal{P}$ with objects $\mathbb{N}$, tensor $m \otimes n = m + n$, and unit 0.

Read n as n **wires**; a morphism $C : n \to m$ as a circuit with n inputs and m outputs.



A PROP makes wire permutations structural

Definition

A **PROP** is a PRO with symmetries $\sigma_{m,n} : m + n \to n + m$.
Diagrammatically, wires may be permuted as structural language.

These are extra **symmetry laws**, on top of the PRO laws from the previous slide.

Unit: $\sigma_{0,n} = \mathrm{id}_n = \sigma_{n,0}$.

$\sigma_{m,n} : m + n \to n + m$

involution

$\sigma_{n,m} \circ \sigma_{m,n} = \mathrm{id}_{m+n}$

naturality

$\sigma \circ (f \otimes \mathrm{id}) = (\mathrm{id} \otimes f) \circ \sigma$

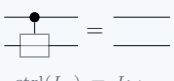
A controlled PROP adds one control constructor

Definition

A **control functor** is an operation $\mathrm{ctrl} : \mathcal{P}_{\mathrm{endo}} \to \mathcal{P}_{\mathrm{endo}}$ with $\mathrm{ctrl}(n) = 1 + n$: it adds one chosen control wire.

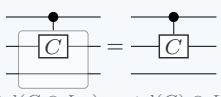
$*$ $\mathcal{P}_{\mathrm{endo}}$: all circuits $n \to n$.

$C : n \to n$
$\mapsto \mathrm{ctrl}(C) : 1 + n \to 1 + n$

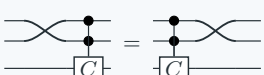


$\mathrm{ctrl}(I_n) = I_{1+n}$

$\mathrm{ctrl}(D \circ C) = \mathrm{ctrl}(D) \circ \mathrm{ctrl}(C)$



$\mathrm{ctrl}(C \otimes I_m) = \mathrm{ctrl}(C) \otimes I_m$



$\mathrm{ctrl}(\mathrm{ctrl}(C)) \circ (\sigma_2 \otimes I)$
$= (\sigma_2 \otimes I) \circ \mathrm{ctrl}(\mathrm{ctrl}(C))$

$\mathrm{ctrl}(\sigma^{-1} \circ C \circ \sigma) = (I \otimes \sigma^{-1}) \circ \mathrm{ctrl}(C) \circ (I \otimes \sigma)$

A polycontrolled PROP adds one constructor per value

Definition

A d-ary **polycontrolled PROP** is a PROP equipped with control functors

$$(\mathrm{ctrl}_k : \mathcal{P}_{\mathrm{endo}} \to \mathcal{P}_{\mathrm{endo}})_{k \in [d]}.$$

Support-sensitivity

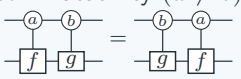
$$\mathrm{supp}(f) \subseteq [d]^n,$$
$$\mathrm{supp}(\mathrm{ctrl}_k(f)) = \{k\} \times \mathrm{supp}(f).$$

Disjoint supports commute.

compatibility

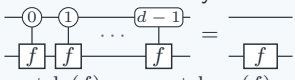
$$\mathrm{ctrl}_a(\mathrm{ctrl}_b(f)) \circ (\sigma_{1,1} \otimes \mathrm{id}_n)$$
$$= (\sigma_{1,1} \otimes \mathrm{id}_n) \circ \mathrm{ctrl}_b(\mathrm{ctrl}_a(f))$$

commutativity $(a \neq b)$



$$\mathrm{ctrl}_a(f) \circ \mathrm{ctrl}_b(g)$$
$$= \mathrm{ctrl}_b(g) \circ \mathrm{ctrl}_a(f)$$

exhaustivity



$$\mathrm{ctrl}_0(f) \circ \cdots \circ \mathrm{ctrl}_{d-1}(f)$$
$$= \mathrm{id}_1 \otimes f$$

Several ways to prove completeness

Normal forms

$$C \iff N(U) \iff D$$
unique

+ canonical representative
+ can become an algorithm
− hard to design for rich syntax
− must prove actual normalization

[Selinger 15] [Li et al. 26]

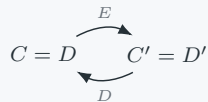
Group presentation

$$\langle G \mid R \rangle \;\cong\; \mathrm{Im}[\![-]\!]$$

+ proves generators and relations
+ good for discrete fragments
− relation sets can explode
− less natural with real parameters

[Bian–Selinger 22] & [23]

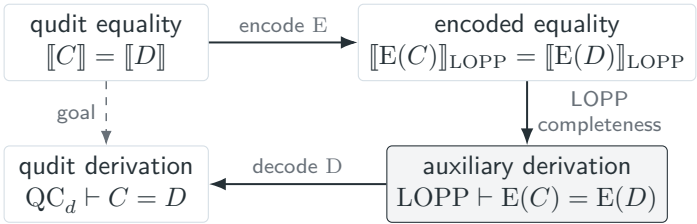
Transfer



+ reuses a complete calculus
+ isolates E, D, and mimicry
− expressivity must match
− auxiliary rules must decode

[Clément et al. 22]

Completeness is transferred by translation



Translation checks

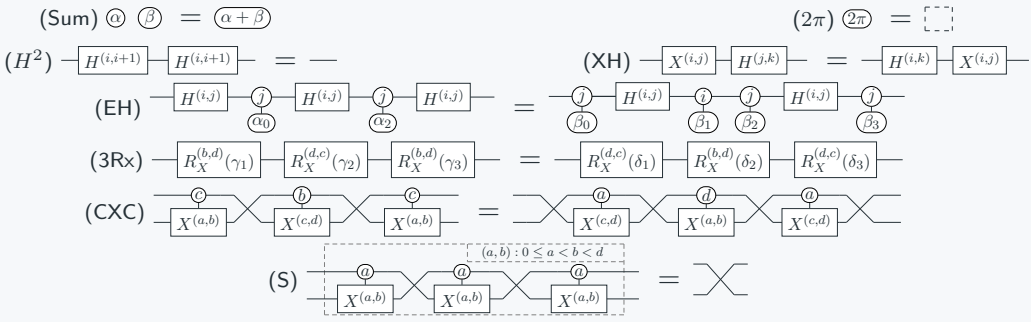
E preserves semantics, $\mathrm{D}(\mathrm{E}(C)) = C$ is derivable, and each LOPP proof step must mimic inside QC_d.

△ LOPP uses $\oplus$, not $\otimes$.

completeness: [Clément et al. 22]

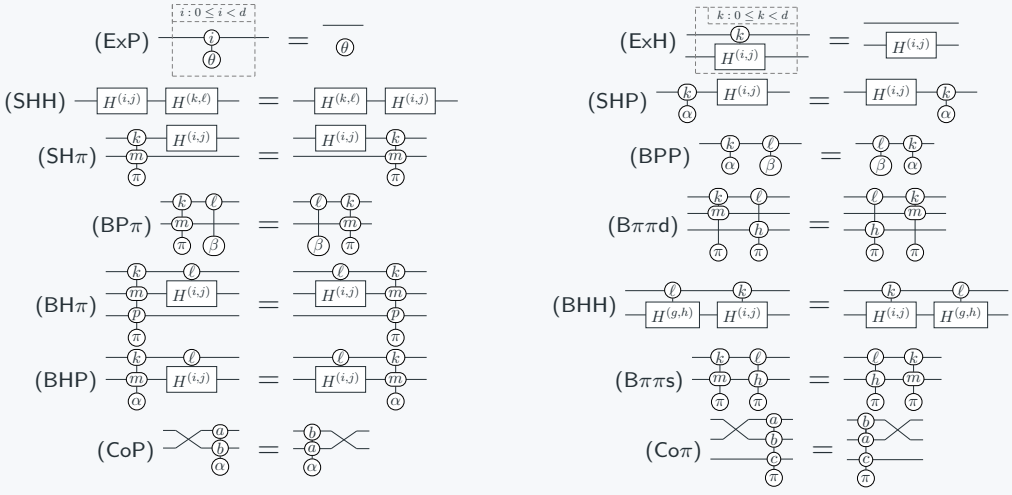
simplified: [Heurtel 25]

Equational theory



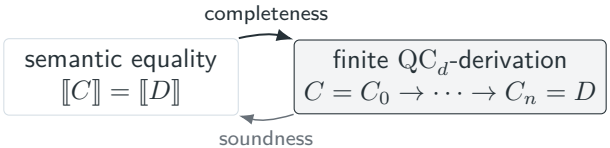
notation: $\mathrm{ctrl}_k(\theta) \equiv \boxed{P_k(\theta)}$

Recovering the properties



Every exact equality has a QC_d-derivation

Theorem. For every finite dimension $d \geq 2$, the finite theory QC_d is **sound and complete** for exact unitary qudit circuits, including global phase.



exact: unitary equality, not approximation or noise

finite: fixed finite schemata for each d

local: bounded circuit windows inside contexts

Implications

- certificates for exact qudit rewrites
- explicit branch syntax
- gate equations separate from control algebra

Future work

- orient fragments as rewrite systems
- native and fault-tolerant gate sets
- arithmetic controls; fragment comparison