Simpler Presentations for Many Fragments of Quantum Circuits

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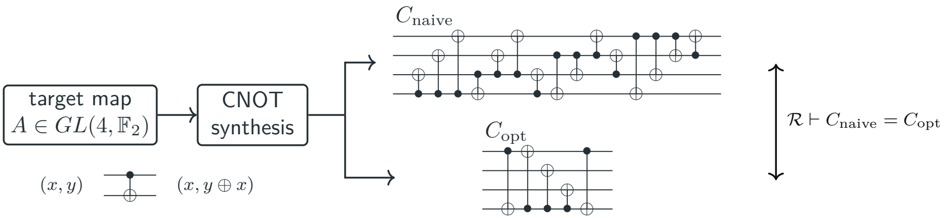
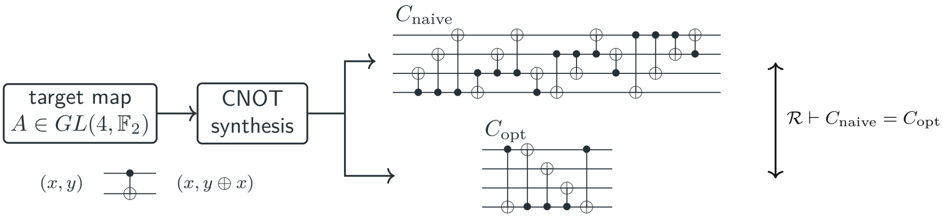
FSCD 2026, ISCTE, Lisbon, Portugal
20–23 July 2026

Completeness proves circuit equivalence



Completeness proves circuit equivalence



Definition (presentation, soundness, completeness)

A presentation $\langle \Sigma \mid \mathcal{R} \rangle$ fixes generators Σ and equations $\mathcal{R}$. For a semantics $[\![-]\!]$, it is *sound* when $\mathcal{R} \vdash C = D \Longrightarrow [\![C]\!] = [\![D]\!]$, and *complete* when $[\![C]\!] = [\![D]\!] \Longrightarrow \mathcal{R} \vdash C = D$.

Ways to prove completeness

Normal forms

$$C \iff \underbrace{N(U)}_{\text{unique}} \iff D$$

+ canonical representative
+ can become an algorithm
− hard to design for rich syntax
− must prove actual normalization

[Selinger 2015] [Li et al. 2025]

Group presentation

$$\langle G \mid R \rangle \;\cong\; \mathrm{Im}[\![-]\!]$$

+ proves generators and relations
+ good for discrete fragments
− relation sets can explode
− less natural with real parameters

[Bian–Selinger 2023a] [2023b]

Transfer

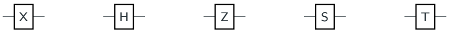
$$\begin{tikzcd} C = D \arrow[r, bend left, "E"] & C' = D' \arrow[l, bend left, "D"] \end{tikzcd}$$

+ reuses a complete calculus
+ isolates E, D, and mimicry
− expressivity must match
− auxiliary rules must decode

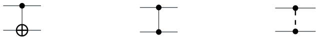
[Clément et al. 2022]

Six circuit fragments

local gates



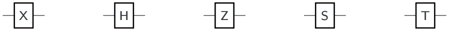
controlled gates



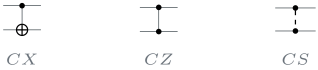
CX CZ CS

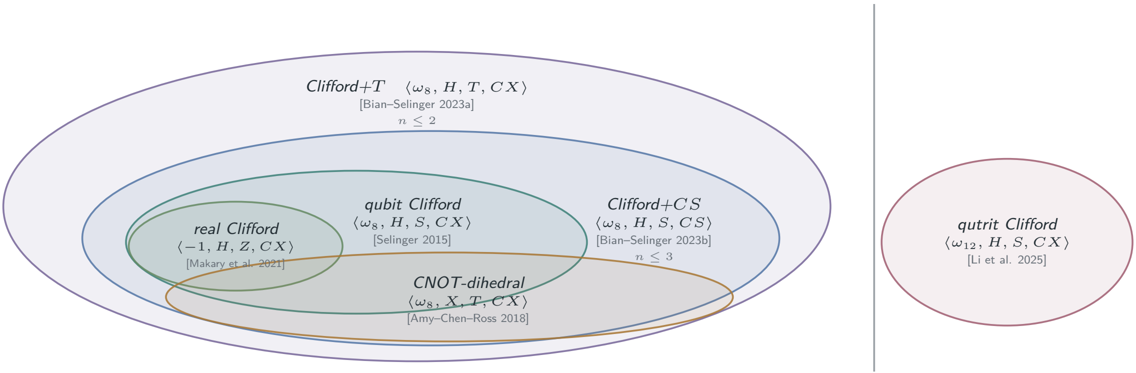
Six circuit fragments

local gates



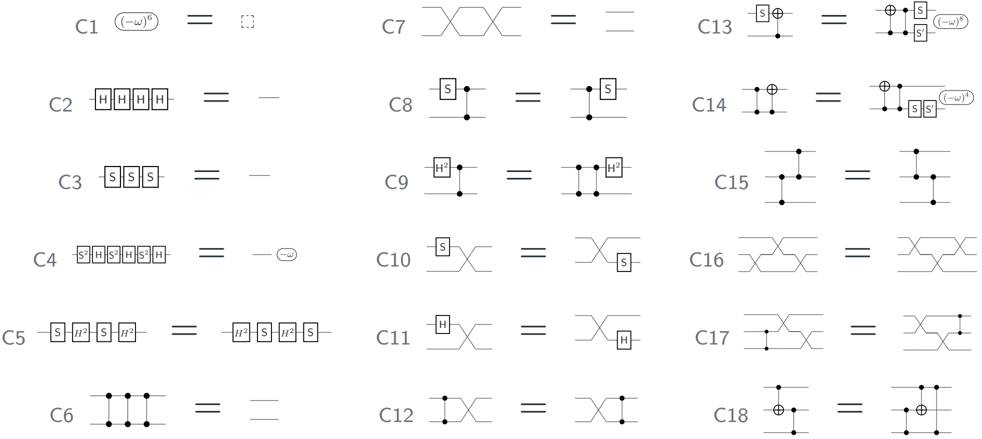
controlled gates





Start with qutrit Clifford

Source complete PRO presentation for qutrit Clifford circuits [Li et al. 2025].



PROs compose circuits

Definition

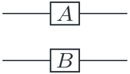
A *PRO* is a strict monoidal category $\mathcal{P}$ with objects $\mathbb{N}$, tensor $m \otimes n = m + n$, and unit 0.

Read n as n *wires*; a morphism $C : n \to m$ as a circuit with n inputs and m outputs.

sequential

$B \circ A$

parallel

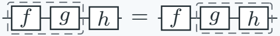


$A \otimes B$

identity

$f \circ \mathrm{id} = f = \mathrm{id} \circ f$

associativity

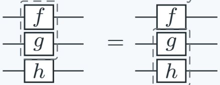


$h \circ (g \circ f) = (h \circ g) \circ f$

tensor unit

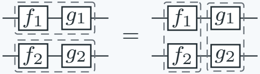
$f \otimes \mathrm{id}_0 = f = \mathrm{id}_0 \otimes f$

tensor associativity



$(f \otimes g) \otimes h = f \otimes (g \otimes h)$

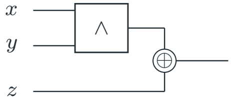
interchange



$(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2)$

PROs model classical networks

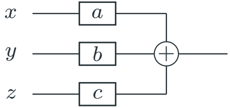
Boolean circuits



$$(x \wedge y) \oplus z$$

[Lafont 2003]

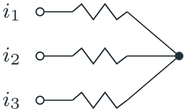
Signal-flow diagrams



$$ax + by + cz$$

[Bonchi–Sobociński–Zanasi 2017]

Electrical networks

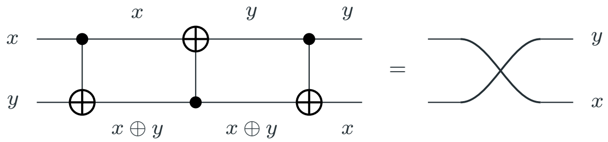


$$i_1 + i_2 + i_3 = 0$$

[Baez–Coya–Rebro 2017]

Qubits hide qutrit corrections

On two qubits, three controlled additions exchange the wires.



Qubits hide qutrit corrections

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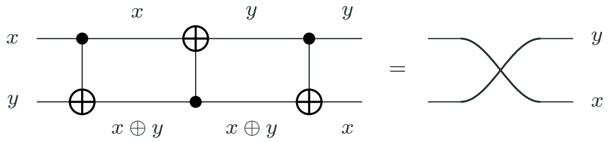
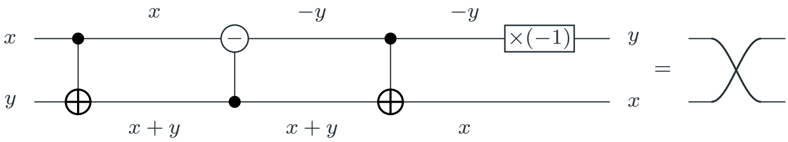


Is it still a swap on qutrits?

Qubits hide qutrit corrections



$\mathbb{Z}_2$: $-u = u$, so both corrections collapse. $\mathbb{Z}_3$: both remain.

PROPs make swaps structural

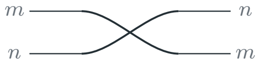
Definition

A *PROP* is a strict symmetric monoidal category $\mathcal{P}$ with

$$\mathrm{Ob}(\mathcal{P}) = \mathbb{N}, \qquad I = 0,$$
$$m \otimes n = m + n.$$

Its symmetry contains a natural family

$$\sigma_{m,n} : m+n \longrightarrow n+m \qquad (m, n \in \mathbb{N})$$



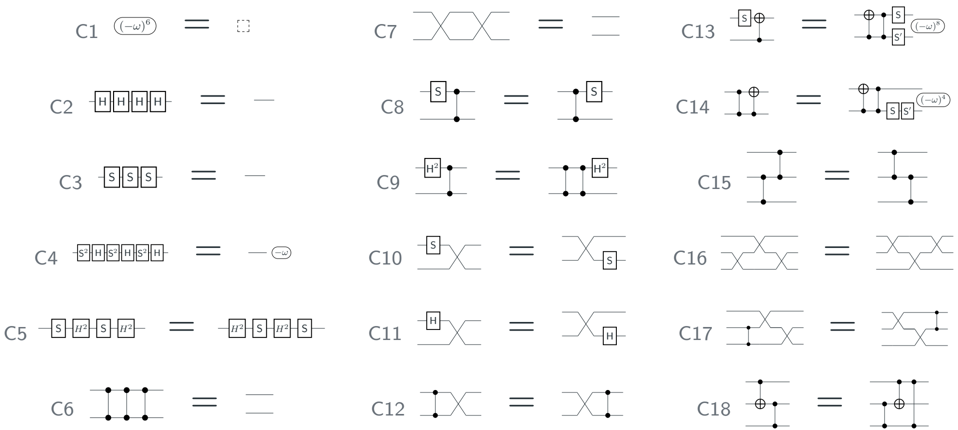
involution

$$\sigma_{n,m} \circ \sigma_{m,n} = \mathrm{id}_{m+n}$$

naturality

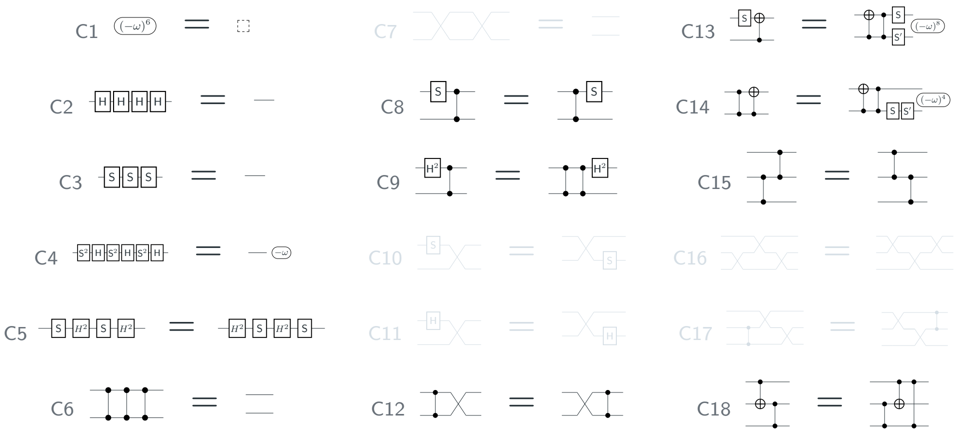
$$\sigma \circ (f \otimes \mathrm{id}) = (\mathrm{id} \otimes f) \circ \sigma$$

Swap equations become structural



The *faded rows* are supplied by *PROP symmetry*: $\sigma^2 = \mathrm{id}$, naturality, and symmetric monoidal coherence.

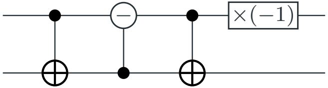
Swap equations become structural



The *faded rows* are supplied by *PROP symmetry*: $\sigma^2 = \mathrm{id}$, naturality, and symmetric monoidal coherence.

One equation transfers completeness

circuit term $\tau : 2 \to 2$ | *structural symmetry* $\sigma_{1,1}$



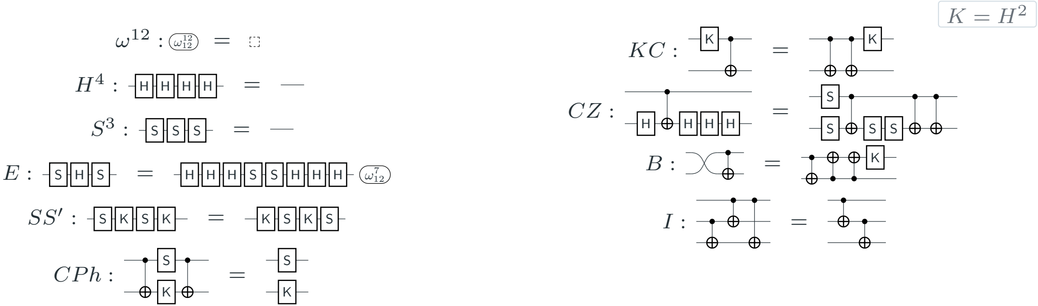
$=$

Completeness transfer

For a PROP term D, let D^τ be the PRO term obtained by replacing each structural crossing by the circuit τ. If $\langle \Sigma \mid \mathcal{R} \rangle_{\mathrm{PRO}}$ is complete and $[\![\tau]\!]$ is the wire swap, then $\langle \Sigma \mid \mathcal{R}, \sigma_{1,1} = \tau \rangle_{\mathrm{PROP}}$ is complete.

$[\![D]\!] = [\![E]\!]$ → $[\![D^\tau]\!] = [\![E^\tau]\!]$ ($\sigma \mapsto \tau$) → $\mathcal{R} \vdash D^\tau = E^\tau$ (source completeness) → $\mathcal{R}, \sigma = \tau \vdash D = E$ (replace τ by σ)

Ten qutrit equations remain



Is each equation necessary?

Independent equation

Let $\rho : L_\rho = R_\rho$ belong to $\mathcal{R}$. Then

$$\rho \text{ is independent} \iff \mathcal{R} \setminus \{\rho\} \nvdash L_\rho = R_\rho.$$

Minimal presentation

$$\mathcal{R} \text{ is minimal} \iff \forall \rho \in \mathcal{R},\ \rho \text{ is independent}.$$

Why check?

- A mechanized proof records each primitive equation as an assumption.
  [Bian–Selinger 2023b]
- Fewer primitive rules reduce branching in derivation search.
- Finite-model search can certify non-derivability.
  [McCune 2003]

Countermodels certify independence

Separation lemma

$$\exists\, [\![\cdot]\!]_{\mathrm{alt}} : \mathcal{P}_\Sigma \longrightarrow \mathcal{M}, \quad [\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R} \setminus \{\rho\}, \quad [\![L_\rho]\!]_{\mathrm{alt}} \neq [\![R_\rho]\!]_{\mathrm{alt}} \quad \Longrightarrow \quad \mathcal{R} \setminus \{\rho\} \nvdash L_\rho = R_\rho.$$

$[\![\cdot]\!]_{\mathrm{alt}}$ is a PROP morphism, so $\ker([\![\cdot]\!]_{\mathrm{alt}})$ is closed under identities, composition, tensor, and structural symmetries.

Equational soundness: [Birkhoff 1935]

A $\mathbb{Z}_4$ countermodel

$$\mathcal{P} = \langle a \mid \rho_2 : a^2 = e,\ \ \rho_4 : a^4 = e\rangle$$

ρ_4 is derivable

The relation ρ_2 gives

$$a^4 = (a^2)^2 = e.$$

Hence ρ_4 is redundant in this presentation.

ρ_2 is independent

Define a monoid homomorphism into $(\mathbb{Z}_4, +)$ by

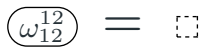
$$F(a^k) = k \bmod 4, \qquad F(e) = 0.$$

$$\begin{aligned} F(a^4) &= 0 = F(e), \\ F(a^2) &= 2 \neq 0 = F(e). \end{aligned}$$

$\{\rho_4\} \nvdash \rho_2$, witnessed by the four-element model $(\mathbb{Z}_4, +)$.

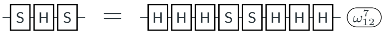
Occurrence separates CPh

In $\mathbb{B}_{\max}$, set $F(\text{CNOT}) = 1$, every other generator and symmetry to 0, and use $\max$ for $\circ$ and $\otimes$.

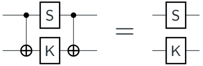
ω^{12}  $0 = 0$

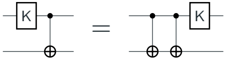
H^4  $0 = 0$

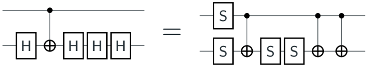
S^3  $0 = 0$

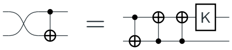
E  $0 = 0$

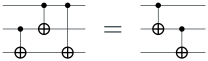
SS'  $0 = 0$

CPh  $1 \neq 0$

KC  $1 = 1$

CZ  $1 = 1$

B  $1 = 1$

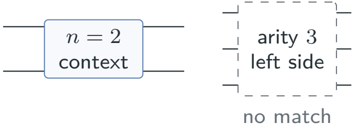
I  $1 = 1$

Arity blocks ternary rules

Arity truncation

If every generator is an endomorphism, then for $C, D : n \to n$,

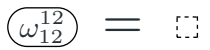
$$\mathcal{R} \vdash C = D \quad \iff \quad \mathcal{R}_{\leq n} \vdash C = D.$$

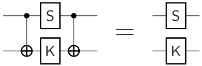
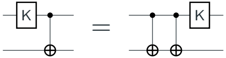
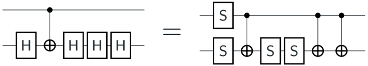
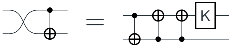
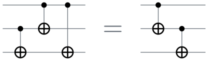


- Every generator has type $k \to k$.
- Composition, tensor, and every rewrite preserve the ambient wire count.
- A rule of arity $k > n$ cannot match inside an n-wire derivation.
- Testing $\rho : n \to n$ therefore requires only $\mathcal{R}_{\leq n}$.
- State preparation, discard, or fresh ancillas invalidate the argument.

Parity distinguishes KC

$[\![\cdot]\!]_{\text{alt}} = \#\{\text{CNOT}\}_{[2]}$, with composition and tensor interpreted as addition in $\mathbb{Z}_2$.

ω^{12}		$0 = 0$
H^4		$0 = 0$
S^3		$0 = 0$
E		$0 = 0$
SS'		$0 = 0$

CPh		$0 = 0$
KC		$1 \neq 0$
CZ		$1 = 1$
B		$1 = 1$
I		$1 \neq 0$

Substitution separates SS'

$$\boxed{\mathsf{S}} \quad \longmapsto \quad \boxed{\mathsf{S}}\boxed{\mathsf{X}}$$

Replace every occurrence of S by SX.

The retained scalar and one-qutrit equations still hold

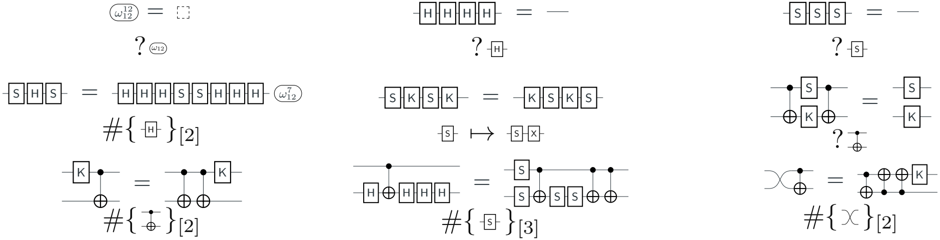
$$\omega^{12} \qquad H^4 \qquad S^3 \qquad E \qquad \checkmark$$

The removed SS' equation fails after substitution

$$\boxed{\mathsf{S}}\boxed{\mathsf{X}}\boxed{\mathsf{K}}\boxed{\mathsf{S}}\boxed{\mathsf{X}}\boxed{\mathsf{K}} \quad \neq \quad \boxed{\mathsf{K}}\boxed{\mathsf{S}}\boxed{\mathsf{X}}\boxed{\mathsf{K}}\boxed{\mathsf{S}}\boxed{\mathsf{X}}$$

Qutrit Clifford: arity two

Each displayed rule has a separating PROP morphism.



Qutrit Clifford: arity two

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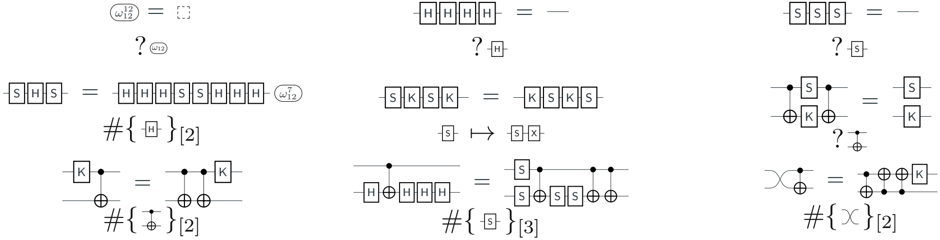
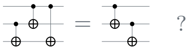


The reduced qutrit presentation is *minimal* for circuits on at most two qutrit wires.



Six reduced presentations

fragment	previous	reduced
qubit Clifford	15	8
real Clifford	16	10
CNOT-dihedral	13	11
Clifford+T ($\leq$ 2q)	18	11
Clifford+CS ($\leq$ 3q)	17	14
qutrit Clifford	18	10

Six reduced presentations

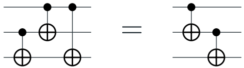
fragment	previous	reduced	minimality status
qubit Clifford	15	8	✓ minimal in all arities
real Clifford	16	10	✓ minimal in all arities
CNOT-dihedral	13	11	✓ minimal in all arities
Clifford+T (≤ 2q)	18	11	○ minimal through 1-qubit circuits; remaining 2-qubit equations undecided
Clifford+CS (≤ 3q)	17	14	○ minimal through 2-qubit circuits; remaining 3-qubit rule family undecided
qutrit Clifford	18	10	○ minimal through 2-qutrits; ternary braid open

Real Clifford + CH

```
PS C:\Users\cblake\Documents\minimality-auto> python -m minimality_auto theories\realclifford-ch-simplified.json --auto --timeout 120
Real-Clifford+CH (Figure 4): 17 separator(s) found
  [found] 19_n5: ambient-5-wire Spin-cover invariant: the commuting target factors lift with sign -1
  [found] 19_n6: ambient-6-wire Spin-cover invariant: the commuting target factors lift with sign -1; 1 retained cyclic reversal(s) scale to sign +1
  [found] 19_n7: ambient-7-wire Spin-cover invariant: the commuting target factors lift with sign -1; 1 retained cyclic reversal(s) scale to sign +1
  [found] 19_n8: ambient-8-wire Spin-cover invariant: the commuting target factors lift with sign -1; 1 retained cyclic reversal(s) scale to sign +1
  [found] 19_n9: ambient-9-wire Spin-cover invariant: the commuting target factors lift with sign -1; 1 retained cyclic reversal(s) scale to sign +1
  [found] 19_n10: ambient-10-wire Spin-cover invariant: the commuting target factors lift with sign -1; 1 retained cyclic reversal(s) scale to sign +1
  [found] 3: discovered F_7 amalgam with bridge {H}: |C|=2, |A|=1, |D|=2
  [found] 11: discovered F_7 amalgam with bridge {CH}: |C|=2304, |A|=256, |D|=512
  [found] 6: count mod 2: {'Z': 1}
  [found] 7: count mod 2: {'H': 1}
  [found] 8: count mod 2: {}, swap=1
  [found] 15: count mod 2: {'CH': 1, 'CZ': 1}, swap=1
  [found] 18: count mod 2: {'CH': 1}
  [found] 1: degree-3 permutation interpretation: generators {H=(1, 2, 0), Z=(0, 2, 1)}
  [found] 2: degree-3 permutation interpretation: generators {H=(0, 2, 1), Z=(1, 2, 0)}
  [found] 10: degree-3 permutation interpretation: generators {H=(0, 1, 2), Z=(0, 1, 2), CZ=(0, 2, 1), CH=(1, 0, 2)}; structural swaps {swap[0,1]=(0, 2, 1)}
  [found] 5: count mod 4: {'CH': 1, 'H': 2}
  [open]  12
  [open]  13
  [open]  14
  [open]  16
  [open]  17
elapsed: 120.000s (timeout)
```

https://github.com/Sukyn/minimality-auto/

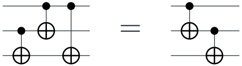
One qutrit braid remains



Conjecture

$$\mathcal{R} \setminus \{I\} \nvdash L_I = R_I.$$

One qutrit braid remains



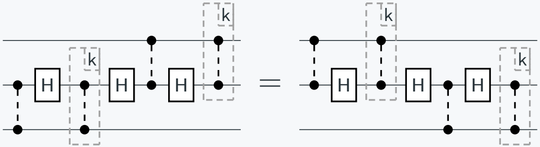
Conjecture

$$\mathcal{R} \setminus \{I\} \nvdash L_I = R_I.$$

- I is the only qutrit rule of arity 3.
- Every qutrit rule of arity at most 2 has an independence certificate.
- The corresponding ternary laws are independent in real Clifford, qubit Clifford, Clifford$+CS$, and CNOT-dihedral.

Four Clifford+CS rules remain

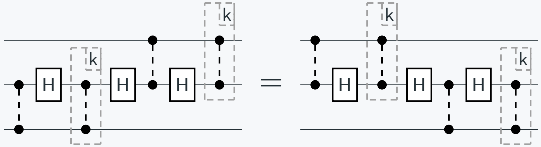
ρ_k :



$k \in \{0, 1, 2, 3\}$

Four Clifford+CS rules remain

ρ_k :



$k \in \{0, 1, 2, 3\}$

Occurrence and counts agree

$$?g(L_k) = ?g(R_k),$$
$$\#\{g\}(L_k) = \#\{g\}(R_k).$$

for the occurrence and counting tests used above.

Substitution target

$$g \mapsto g'$$
$$[\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R} \setminus \{\rho_k\}$$
$$[\![L_k]\!]_{\mathrm{alt}} \neq [\![R_k]\!]_{\mathrm{alt}}.$$

A separator for ρ_k would satisfy both conditions.

What must separators preserve?

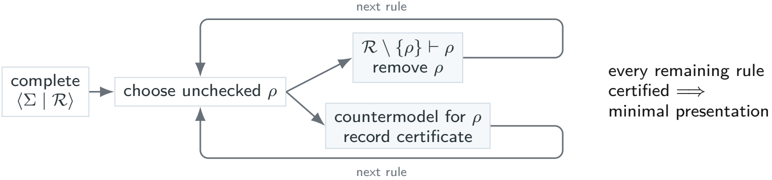
circuit language	structural operations in derivations	requirement on F
PRO	identity, $\circ$, $\otimes$	strict monoidal functor
PROP	PRO operations and σ	preserve canonical symmetry
controlled PROP	PROP operations and control	preserve the control operation
dagger / projective calculus	dagger or scalar quotient	preserve $\dagger$ or projective equality

Independence test

$$F \models \mathcal{R} \setminus \{\rho\} \qquad F(L_\rho) \neq F(R_\rho).$$

The kernel of F must be closed under every context available to a derivation.

Conclusion



Results

- Six complete presentations share one PROP formulation.
- Real Clifford, qubit Clifford, and CNOT-dihedral are minimal.
- Qutrit Clifford is minimal through arity 2.

Future work

- Complete unbounded Clifford+T and Clifford+CS.
- Automate derivation and countermodel search.
- Export certificates to Lean, Agda, or Rocq.

Backup 1: separation argument

Lemma
Let $\mathcal{P}_\Sigma$ be the free PROP on the fragment signature, let $\mathcal{R}$ be a finite set of equations in $\mathcal{P}_\Sigma$, and let $\rho : L = R \in \mathcal{R}$. Put $\mathcal{S} = \mathcal{R} \setminus \{\rho\}$. Suppose there is a PROP M and a strict symmetric monoidal functor $F : \mathcal{P}_\Sigma \to M$, identity on objects, such that

$$\forall (A = B) \in \mathcal{S}, \quad F(A) = F(B), \qquad F(L) \neq F(R).$$

Then $\mathcal{S} \nvdash L = R$, so ρ is independent.

Proof
Define

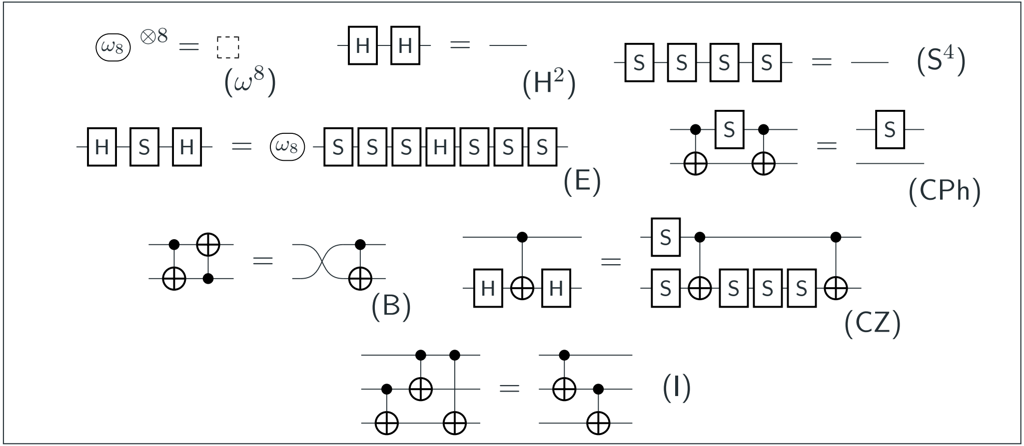
$$E_F = \{(A, B) \mid F(A) = F(B)\}.$$

Since F preserves identities, symmetries, tensor, and sequential composition, E_F is a PROP congruence on $\mathcal{P}_\Sigma$. Since $F \models \mathcal{S}$, every equation of $\mathcal{S}$ lies in E_F. The derivability relation generated by $\mathcal{S}$ is the least PROP congruence containing $\mathcal{S}$, hence

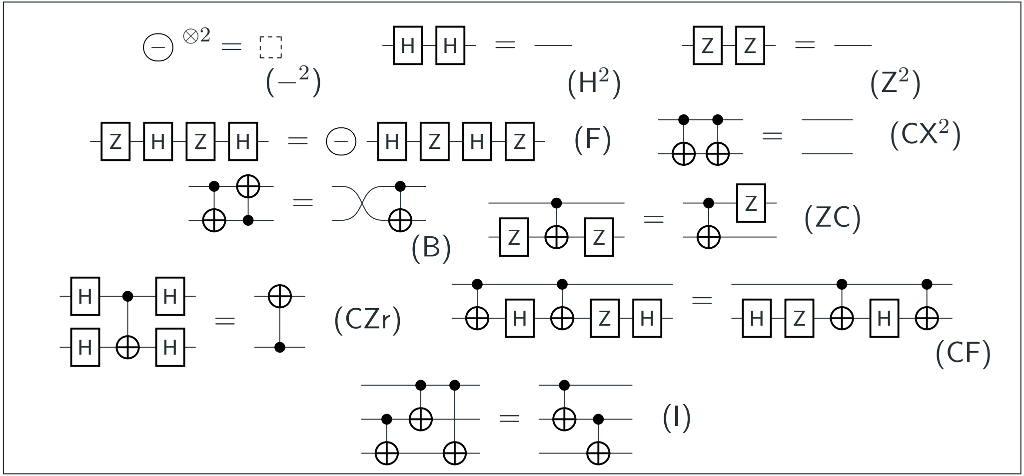
$$\mathcal{S} \vdash A = B \quad \Longrightarrow \quad F(A) = F(B).$$

Applying this to $A = L$, $B = R$ would contradict $F(L) \neq F(R)$.
Countermodel/soundness direction of the Birkhoff-style argument.

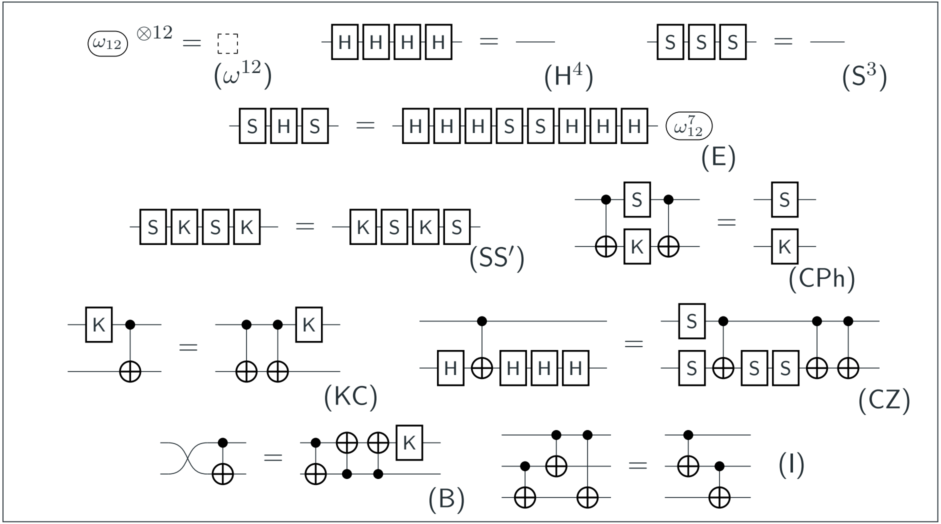
Backup 2: qubit Clifford



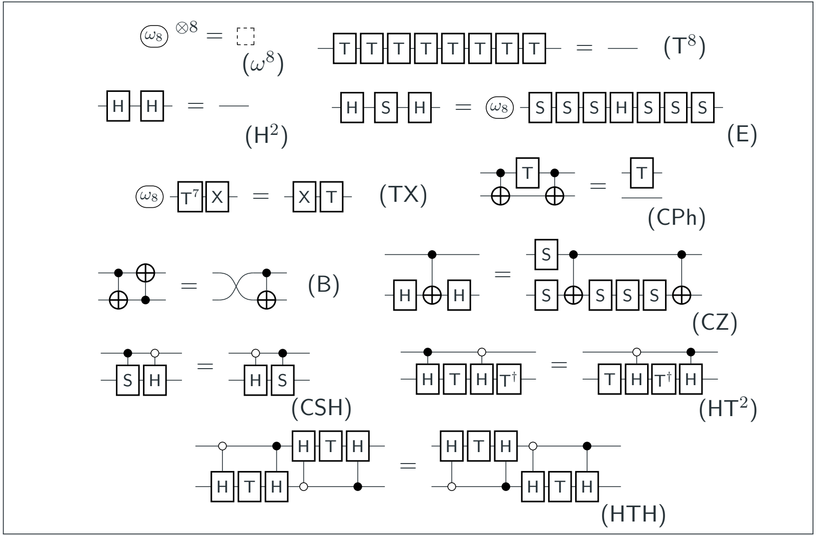
Backup 3: real Clifford



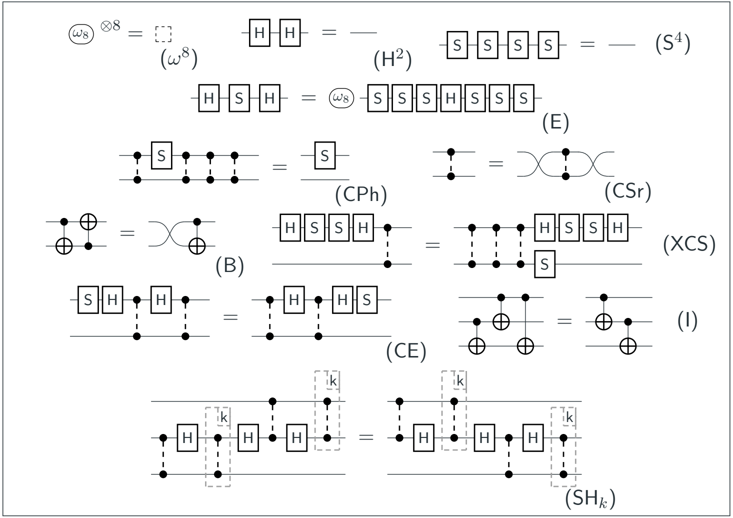
Backup 4: qutrit Clifford



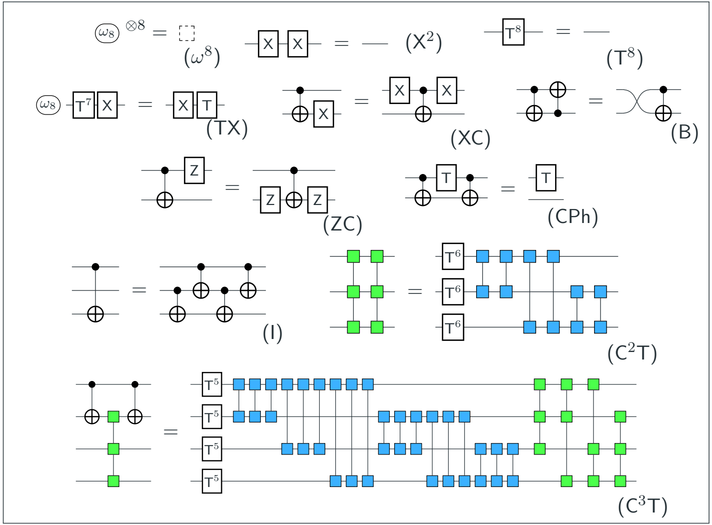
Backup 5: Clifford+T relations



Backup 6: Clifford+CS relations



Backup 7: CNOT-dihedral relations



Backup 8: separators I

qubit Clifford

relation	separator
ω^8	$?\omega_8$
H^2	$?H$
S^4	$?S$
E	$\#\{H\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
CZ	$\#\{CX, SWAP\}_{[2]}$
I	$\arg\det_2$

real Clifford

relation	separator
$-^2$	$?-$
H^2	$?H$
Z^2	$?Z$
F	$\#\{\omega\}_{[2]}$
CX^2	$?CX$
B	$\#\{SWAP\}_{[2]}$
ZC	$\#\{Z\}_{[2]}$
CF	$Z \mapsto I$
CZr	$H \mapsto I$
I	$\arg\det_2$

qutrit Clifford

relation	separator
ω^{12}	$?\omega_{12}$
H^4	$?H$
S^3	$?S$
E	$\#\{H\}_{[2]}$
SS'	$S \mapsto SX$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
CZ	$\#\{S\}_{[3]}$
KC	$\#\{CX\}_{[2]}$
I	–

Backup 9: separators II

Clifford+T

relation	separator
ω^8	$?\omega_8$
H^2	$?H$
T^8	$?T$
E	$\#\{H\}_{[2]}$
TX	$\#\{H, T, \omega\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
CZ	$\#\{CX, SWAP\}_{[2]}$
CSH	–
HT^2	–
HTH	–

Clifford+CS

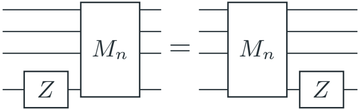
relation	separator
ω^8	$?\omega_8$
H^2	$?H$
S^4	$?S$
E	$\#\{H\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
XCS	$\#\{S, H\}_{[2]}$
CSr	$CS \mapsto CS_z$
CE	$CS \mapsto CS_{zz}$
I	$\arg\det_2$
SH_k	–

CNOT-dihedral

relation	separator
ω^8	$?\omega_8$
T^8	$?T$
X^2	$?X$
TX	$\#\{\omega\}_{[2]}$
XC	$\#\{X\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
ZC	$\#\{T, \omega, \omega\}_{[8]}$
I	$\#\{CX, SWAP\}_{[2]}$
C^2T	$\#\{T, \omega, \omega\}_{[4]}$
C^3T	$\arg\det_3$

Backup 10: Spin separation

Rule $(19)_n$, $n \geq 5$



$$M_n = \mathrm{MCZ}_{n-1} \otimes I$$

$$\begin{tikzcd} & \mathrm{Spin}(2^n) & \\ u \arrow[dr] & & -u \arrow[dl] \\ & A \in SO(2^n) & \end{tikzcd}$$

The matrix forgets the central sign.

For commuting orthogonal involutions,

$$\widehat{A}\widehat{B} = (-1)^{pq-r}\widehat{B}\widehat{A}.$$

$$p = \dim E_-(Z) = 2^{n-1},$$
$$q = \dim E_-(M_n) = 2,$$
$$r = \dim\big(E_-(Z) \cap E_-(M_n)\big) = 1.$$
$$pq - r = 2^n - 1 \equiv 1 \pmod 2.$$

$$\widehat{Z}\widehat{M_n} = -\widehat{M_n}\widehat{Z}.$$

Rules (1)–(18)

$m \leq 3 \implies K = 2^{n-m} \implies 4 \mid K \ (n \geq 5) \implies$ Spin sign $+1$