Completeness Is Not Enough

Simpler presentations and minimality for near-Clifford circuit fragments

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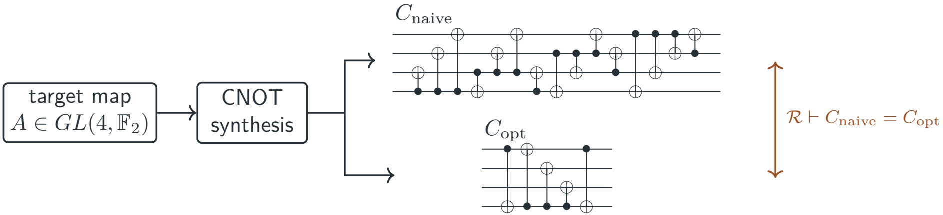
Completeness proves circuit equivalence

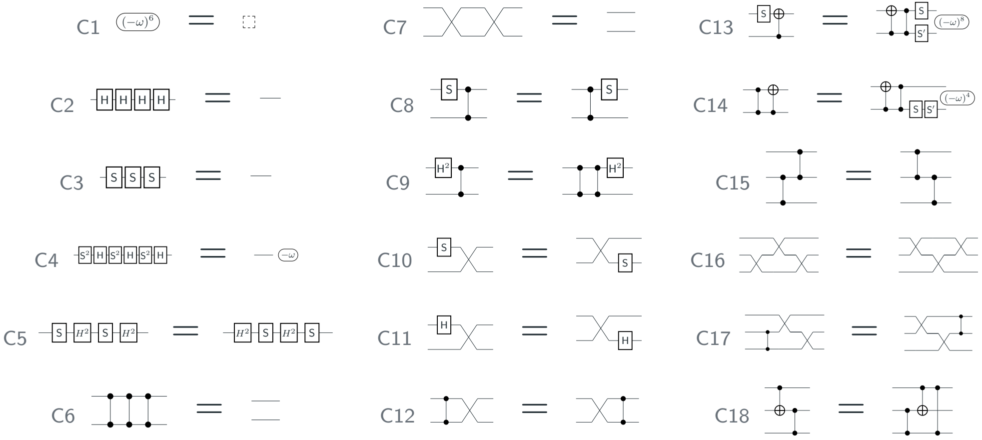


Definition (Presentation, soundness, completeness)

A presentation $\langle \Sigma \mid \mathcal{R} \rangle$ fixes generators Σ and equations $\mathcal{R}$. For a semantics $[\![-]\!]$, it is *sound* when $\mathcal{R} \vdash C = D \implies [\![C]\!] = [\![D]\!]$, and *complete* when $[\![C]\!] = [\![D]\!] \implies \mathcal{R} \vdash C = D$.

Start with qutrit Clifford

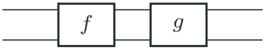
Source complete PRO presentation for qutrit Clifford circuits (Li et al., QPL 2025).



PRO: composition is structural

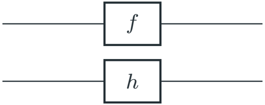
Definition (PRO)
A PRO is a strict monoidal category with objects $\mathbb{N}$, tensor $m \otimes n = m + n$, and monoidal unit 0.

sequential composition



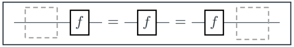
$g \circ f : 2 \to 2$

parallel composition



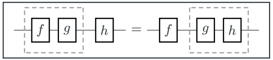
$f \otimes h : 2 \to 2$

PRO coherence laws



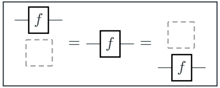
identity for ∘

$f \circ \mathrm{id} = \mathrm{id} \circ f = f$



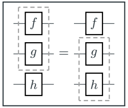
associativity of ∘

$(h \circ g) \circ f = h \circ (g \circ f)$



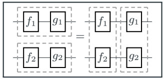
unit for ⊗

$f \otimes \mathrm{id}_0 = \mathrm{id}_0 \otimes f = f$



associativity of ⊗

$(f \otimes g) \otimes h = f \otimes (g \otimes h)$



interchange

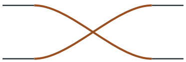
$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$

PROP: symmetry is structural

Definition (PROP)

A PROP is a strict symmetric monoidal category with objects $\mathbb{N}$, $m \otimes n = m + n$, and symmetries $\sigma_{m,n} : m + n \to n + m$.

$\sigma_{1,1} : 2 \to 2$

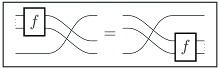


wire permutations are part of the structure

PROP symmetry laws

$\sigma^2 = \mathrm{id}_2$

$\sigma_{1,1} \circ \sigma_{1,1} = \mathrm{id}_2$

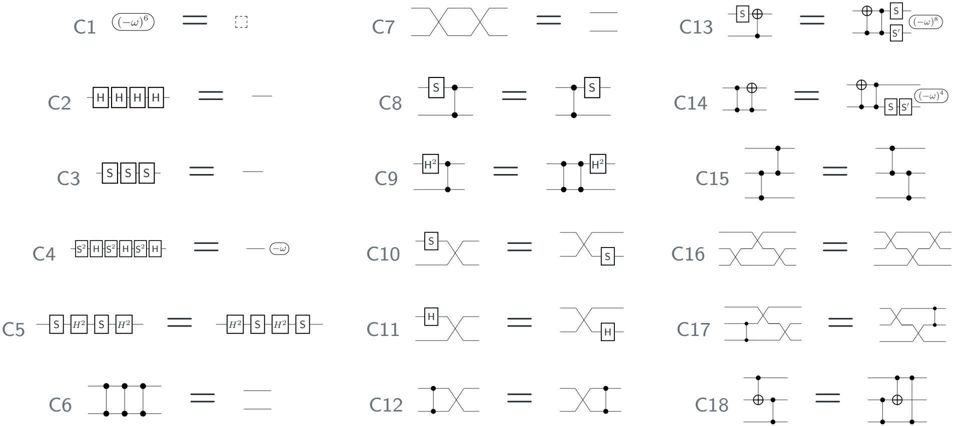


naturality of σ

$\sigma \circ (f \otimes \mathrm{id}) = (\mathrm{id} \otimes f) \circ \sigma$

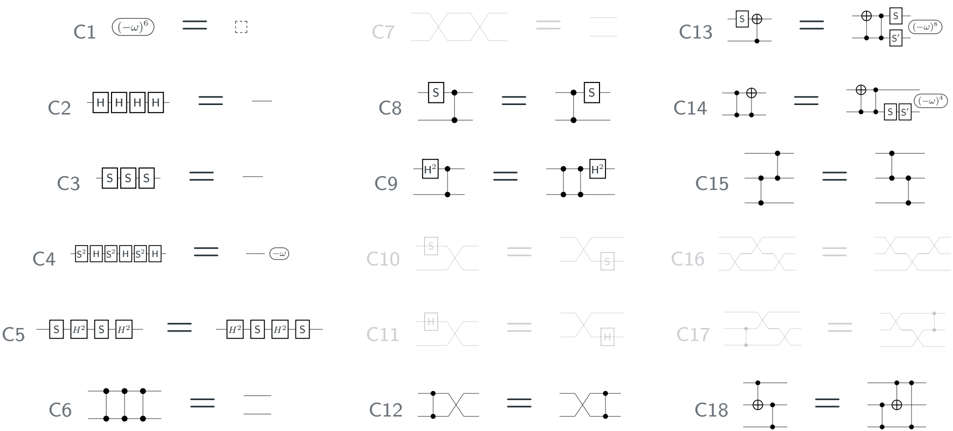
When an old equation only manages wires, it becomes an equation of the ambient diagrammatic language.

Some equations become ambient



The *faded rows* are supplied by *PROP symmetry*: $\sigma^2 = \mathrm{id}$, naturality, and symmetric monoidal coherence.

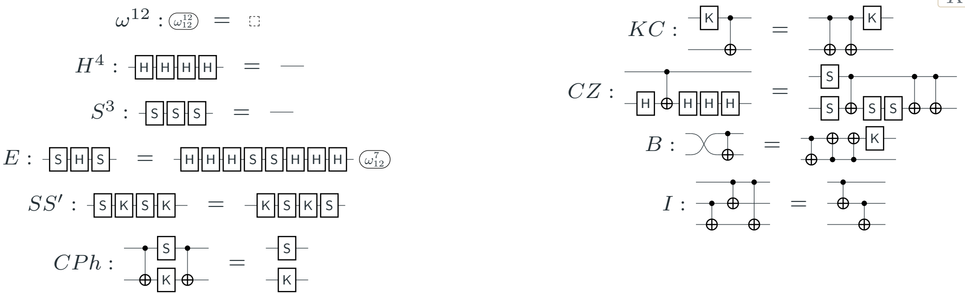
Some equations become ambient



The *faded rows* are supplied by *PROP symmetry*: $\sigma^2 = \mathrm{id}$, naturality, and symmetric monoidal coherence.

The reduced qutrit PROP presentation

$K = H^2$



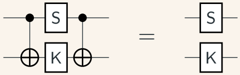
The 18 source PRO relations become 10 *non-structural* PROP relations, still *complete* for qutrit Clifford circuits.

Minimality tests one removed rule

Pick a circuit equation. Remove it.

Example removal test

ρ :



$$\mathcal{R}_{\mathrm{red}} \setminus \{\rho\} \ \vdash \ L_\rho = R_\rho \quad ?$$

Definition
An axiom $\rho : L = R \in \mathcal{R}$ is *independent* when

$$\mathcal{R} \setminus \{\rho\} \nvdash L = R.$$

A finite presentation is *minimal* when every axiom is independent.

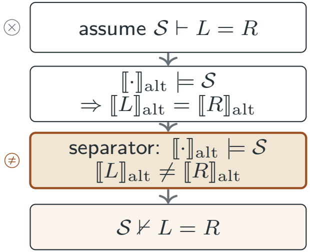
Here $\mathcal{S} \vdash L = R$ means equality generated by $\mathcal{S}$ under composition, tensor, and symmetry.

Separation proves independence

Lemma (Separation)
Let $\rho : L = R \in \mathcal{R}$, and remove it from the presentation. If there is an alternative PROP interpretation $[\![\cdot]\!]_{\mathrm{alt}}$ such that

$$[\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R} \setminus \{\rho\} \quad \text{and} \quad [\![L]\!]_{\mathrm{alt}} \neq [\![R]\!]_{\mathrm{alt}},$$

then $\mathcal{R} \setminus \{\rho\} \nvdash L = R$. Hence ρ is *independent*.



$\mathcal{S} = \mathcal{R} \setminus \{\rho\}$

$$\exists\, [\![\cdot]\!]_{\mathrm{alt}} \left([\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R} \setminus \{\rho\} \wedge [\![L]\!]_{\mathrm{alt}} \neq [\![R]\!]_{\mathrm{alt}}\right) \\ \implies \mathcal{R} \setminus \{\rho\} \nvdash L = R.$$

Occurrence separates a binary equation

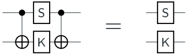
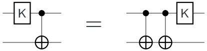
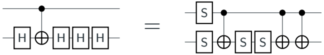
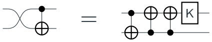
Define $[\![\cdot]\!]_{\mathrm{alt}} =?$  .

It returns 1 exactly when the  generator occurs.

scalar and one-qutrit equations

ω^{12}		$[\![\cdot]\!]_{\mathrm{alt}} : 0 = 0$
H^4		$[\![\cdot]\!]_{\mathrm{alt}} : 0 = 0$
S^3		$[\![\cdot]\!]_{\mathrm{alt}} : 0 = 0$
E		$[\![\cdot]\!]_{\mathrm{alt}} : 0 = 0$
SS'		$[\![\cdot]\!]_{\mathrm{alt}} : 0 = 0$

interaction equations

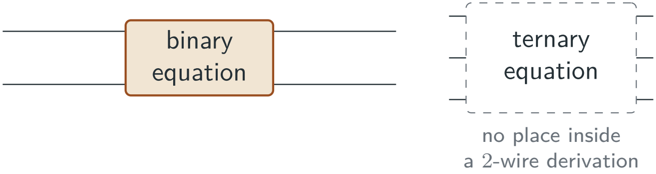
CPh		$[\![\cdot]\!]_{\mathrm{alt}} : 1 \neq 0$
KC		$[\![\cdot]\!]_{\mathrm{alt}} : 1 = 1$
CZ		$[\![\cdot]\!]_{\mathrm{alt}} : 1 = 1$
B		$[\![\cdot]\!]_{\mathrm{alt}} : 1 = 1$
I		$[\![\cdot]\!]_{\mathrm{alt}} : 1 = 1$

$\rho : CPh \quad [\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R} \setminus \{CPh\}$ and $[\![L_{CPh}]\!]_{\mathrm{alt}} \neq [\![R_{CPh}]\!]_{\mathrm{alt}}$. Therefore CPh is independent.

A binary rule only needs binary checks

Lemma (arity truncation). Suppose every generator in a presentation is an endomorphism. Let $\mathcal{R}_{\leq n}$ be the equations in $\mathcal{R}$ whose sides have arity at most n. For $C, D : n \to n$,

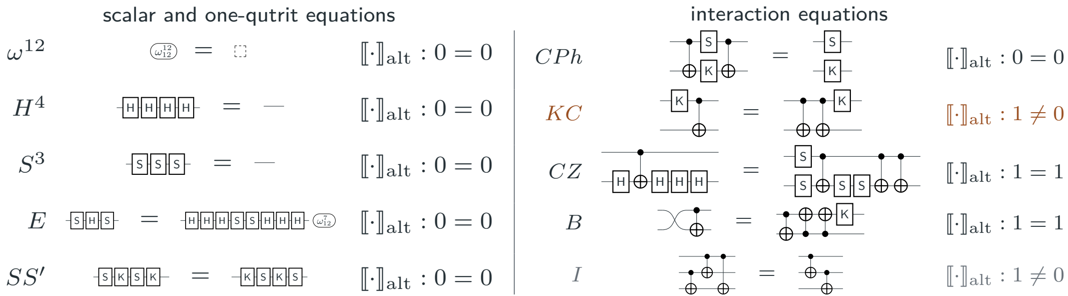
$$\mathcal{R} \vdash C = D \quad \Longleftrightarrow \quad \mathcal{R}_{\leq n} \vdash C = D.$$



The presentation is complete at all arities; for a 2-wire equation, an independence proof only has to validate the retained equations of arity 0, 1, and 2.

CX parity uses the arity lemma

For the KC equation, use $[\![\cdot]\!]_{\mathrm{alt}} = \#\{\text{CX}\}_{[2]}$ on the 0-, 1-, and 2-wire sub-PROP.



By arity truncation, the ternary braid is outside the proof obligation for the binary rule $KC : 2 \to 2$.

A substitution separates a one-qutrit equation

Define $[\![\cdot]\!]_{\mathrm{alt}}$ by $-\boxed{S}- \;\mapsto\; -\boxed{S}-\boxed{X}-$.

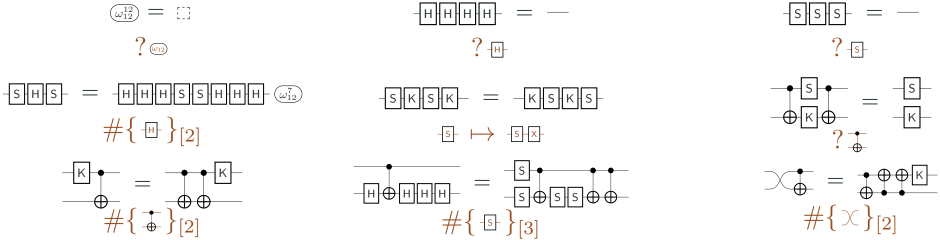
The generators ω_{12}, H, and  keep their usual projective qutrit interpretation.

ω^{12}	$(\omega_{12}^{12}) = \square$	unchanged
H^4	$-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}- \;\simeq\; -$	H unchanged
S^3	$-\boxed{S}-\boxed{X}-\boxed{S}-\boxed{X}-\boxed{S}-\boxed{X}- \;\simeq\; -$	image of $S^3 = 1$
E	$-\boxed{S}-\boxed{X}-\boxed{H}-\boxed{S}-\boxed{X}- \;\simeq\; -\boxed{H^3}-\boxed{S}-\boxed{X}-\boxed{S}-\boxed{X}-\boxed{H^3}- \;(\omega_{12}^{3})$	image of the H, S relation
SS'	$-\boxed{S}-\boxed{X}-\boxed{K}-\boxed{S}-\boxed{X}-\boxed{K}- \;\not\simeq\; -\boxed{K}-\boxed{S}-\boxed{X}-\boxed{K}-\boxed{S}-\boxed{X}-$	separated

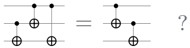
$[\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R}_{\leq 1} \setminus \{SS'\}$, but the two sides of the SS' equation have different projective images.

Qutrit Clifford is settled through two wires

Each displayed rule has a separating PROP morphism.



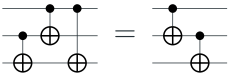
The reduced qutrit presentation is *minimal* on the full sub-PROP generated by the objects 0, 1, and 2.



Reduced presentations across six fragments

fragment	previous	reduced	minimality status
qubit Clifford	15	8	✓ minimal in all arities
real Clifford	16	10	✓ minimal in all arities
CNOT-dihedral	13	11	✓ minimal in all arities
Clifford+T ($\leq$ 2q)	18	11	◯ minimal through 1-qubit circuits; remaining 2-qubit equations undecided
Clifford+CS ($\leq$ 3q)	17	14	◯ minimal through 2-qubit circuits; remaining 3-qubit rule family undecided
qutrit Clifford	18	10	◯ minimal through 2-qutrits; ternary braid open

One ternary braid remains



Conjecture

The reduced qutrit Clifford PROP presentation is minimal. Equivalently, for the ternary braid $I : L_I = R_I$,

$$\mathcal{R} \setminus \{I\} \nvdash L_I = R_I.$$

Proven so far:

$$\forall \rho \in \mathcal{R}_{\leq 2}, \quad \mathcal{R} \setminus \{\rho\} \nvdash \rho.$$

The displayed 3-wire equation is the remaining case.

Can we use other separators?

A candidate separator is a PROP morphism that validates all retained equations and falsifies the tested one.

Structure

$[\![\cdot]\!]_{\mathrm{alt}} : \mathcal{P}_\Sigma \to M$, with M a PROP

$[\![\sigma]\!]_{\mathrm{alt}} = \sigma_M$

$[\![f \otimes g]\!]_{\mathrm{alt}} = [\![f]\!]_{\mathrm{alt}} \otimes [\![g]\!]_{\mathrm{alt}}$

$[\![g \circ f]\!]_{\mathrm{alt}} = [\![g]\!]_{\mathrm{alt}} \circ [\![f]\!]_{\mathrm{alt}}$

Equations

$[\![L_\alpha]\!]_{\mathrm{alt}} = [\![R_\alpha]\!]_{\mathrm{alt}}$ for every $\alpha \neq \rho$

$[\![L_\rho]\!]_{\mathrm{alt}} \neq [\![R_\rho]\!]_{\mathrm{alt}}$

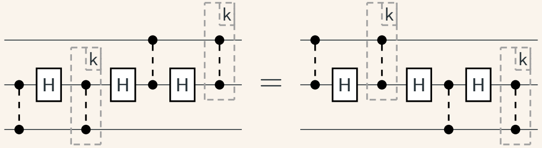
hence $\mathcal{R} \setminus \{\rho\} \nvdash L_\rho = R_\rho$

Target families searched

$?g$ | $\#\{g\} \bmod n$ | $(\#g_1, \ldots, \#g_r)$ | $g \mapsto g'$ | $\mathbf{PU}(d)$ | finite PROPs | etc...

Clifford+CS: one template, four open rules

ρ_k :



$k \in \{0, 1, 2, 3\}$

Occurrence and counts agree

$$?g(L_k) = ?g(R_k),$$
$$\#\{g\}(L_k) = \#\{g\}(R_k).$$

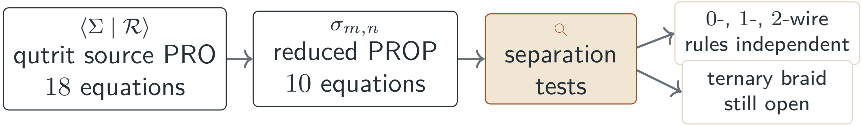
for the occurrence and counting tests used above.

Substitution target

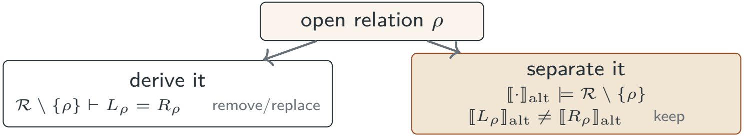
$$g \mapsto g'$$
$$[\![\cdot]\!]_{\mathrm{alt}} \models \mathcal{R} \setminus \{\rho_k\}$$
$$[\![L_k]\!]_{\mathrm{alt}} \neq [\![R_k]\!]_{\mathrm{alt}}.$$

A separator for ρ_k would satisfy both conditions.

Complete, then independent



For every open row $\rho : L_\rho = R_\rho$, prove one of:



Questions?

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Backup 1: formal separation argument

Lemma
Let $\mathcal{P}_\Sigma$ be the free PROP on the fragment signature, let $\mathcal{R}$ be a finite set of equations in $\mathcal{P}_\Sigma$, and let $\rho : L = R \in \mathcal{R}$. Put $\mathcal{S} = \mathcal{R} \setminus \{\rho\}$. Suppose there is a PROP M and a strict symmetric monoidal functor $F : \mathcal{P}_\Sigma \to M$, identity on objects, such that

$$\forall (A = B) \in \mathcal{S}, \quad F(A) = F(B), \qquad F(L) \neq F(R).$$

Then $\mathcal{S} \nvdash L = R$, so ρ is independent.

Proof
Define

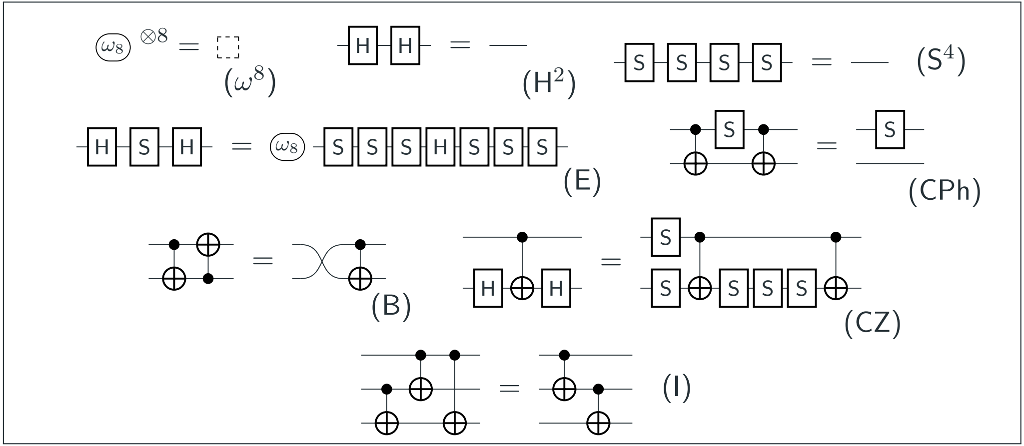
$$E_F = \{(A, B) \mid F(A) = F(B)\}.$$

Since F preserves identities, symmetries, tensor, and sequential composition, E_F is a PROP congruence on $\mathcal{P}_\Sigma$. Since $F \models \mathcal{S}$, every equation of $\mathcal{S}$ lies in E_F. The derivability relation generated by $\mathcal{S}$ is the least PROP congruence containing $\mathcal{S}$, hence

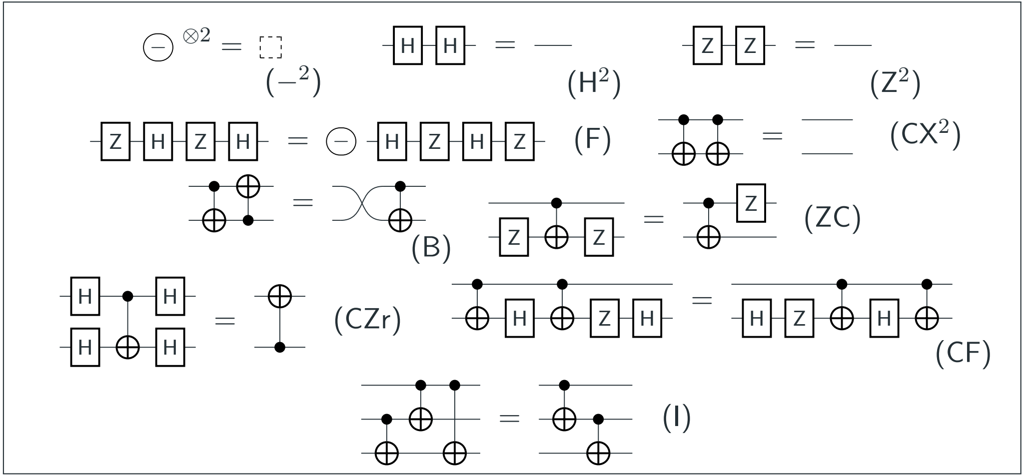
$$\mathcal{S} \vdash A = B \quad \Longrightarrow \quad F(A) = F(B).$$

Applying this to $A = L$, $B = R$ would contradict $F(L) \neq F(R)$.
Countermodel/soundness direction of the Birkhoff-style argument.

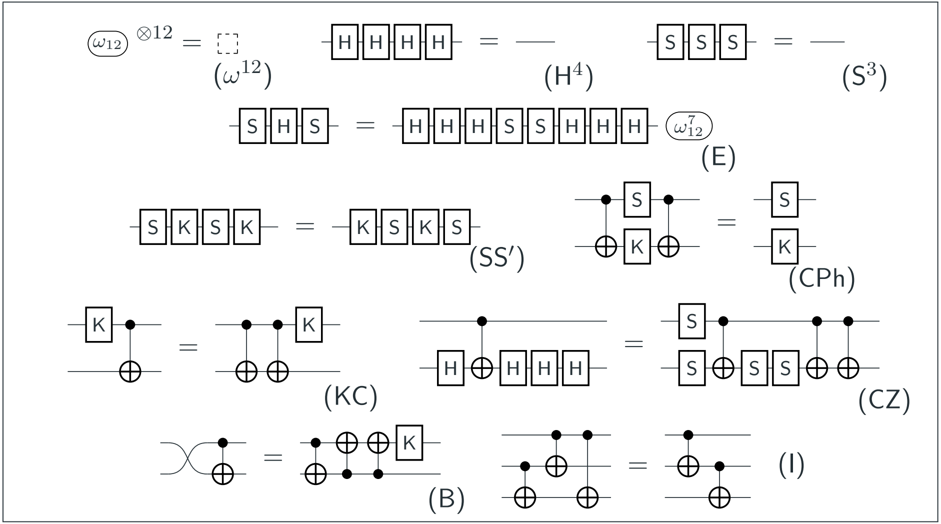
Backup 2: qubit Clifford relations



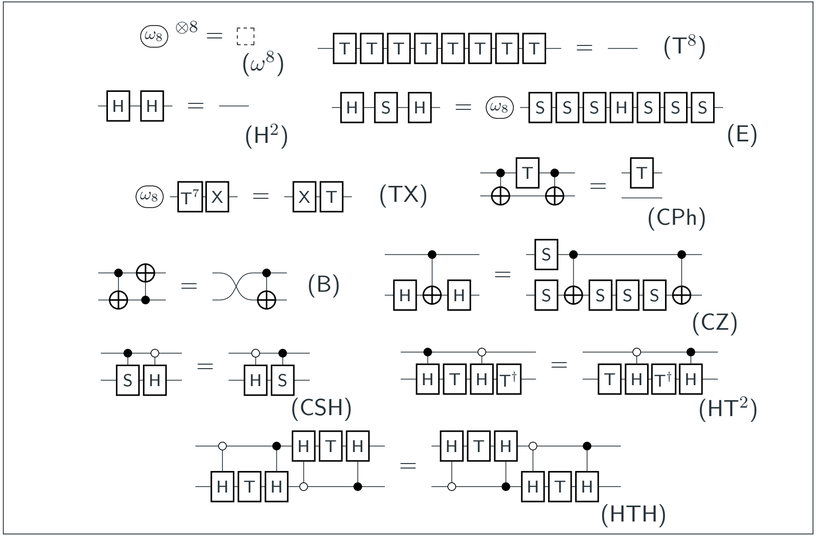
Backup 3: real Clifford relations



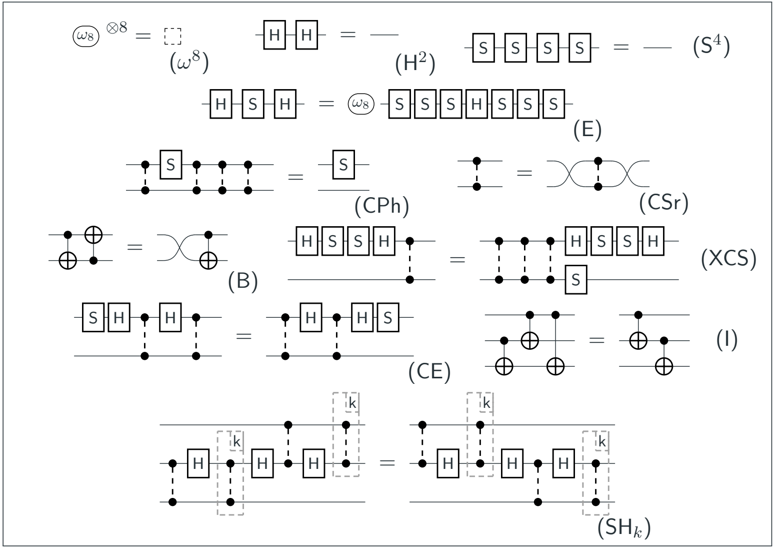
Backup 4: qutrit Clifford relations



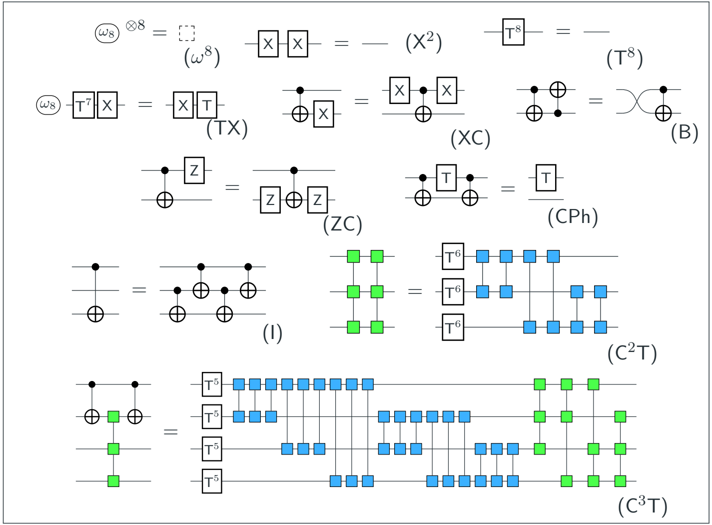
Backup 5: Clifford+T relations



Backup 6: Clifford+CS relations



Backup 7: CNOT-dihedral relations



Backup 8: minimality separators I

qubit Clifford

relation	separator
ω^8	$?\omega_8$
H^2	$?H$
S^4	$?S$
E	$\#\{H\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
CZ	$\#\{CX, SWAP\}_{[2]}$
I	$\arg\det_2$

real Clifford

relation	separator
$-^2$	$?-$
H^2	$?H$
Z^2	$?Z$
F	$\#\{\omega\}_{[2]}$
CX^2	$?CX$
B	$\#\{SWAP\}_{[2]}$
ZC	$\#\{Z\}_{[2]}$
CF	$Z \mapsto I$
CZr	$H \mapsto I$
I	$\arg\det_2$

qutrit Clifford

relation	separator
ω^{12}	$?\omega_{12}$
H^4	$?H$
S^3	$?S$
E	$\#\{H\}_{[2]}$
SS'	$S \mapsto SX$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
CZ	$\#\{S\}_{[3]}$
KC	$\#\{CX\}_{[2]}$
I	–

Backup 9: minimality separators II

Clifford+T	
relation	separator
ω^8	$?\omega_8$
H^2	$?H$
T^8	$?T$
E	$\#\{H\}_{[2]}$
TX	$\#\{H, T, \omega\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
CZ	$\#\{CX, SWAP\}_{[2]}$
CSH	–
HT^2	–
HTH	–

Clifford+CS	
relation	separator
ω^8	$?\omega_8$
H^2	$?H$
S^4	$?S$
E	$\#\{H\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
XCS	$\#\{S, H\}_{[2]}$
CSr	$CS \mapsto CS_z$
CE	$CS \mapsto CS_{zz}$
I	$\arg\det_2$
SH_k	–

CNOT-dihedral	
relation	separator
ω^8	$?\omega_8$
T^8	$?T$
X^2	$?X$
TX	$\#\{\omega\}_{[2]}$
XC	$\#\{X\}_{[2]}$
CPh	$?CX$
B	$\#\{SWAP\}_{[2]}$
ZC	$\#\{T, \omega, \omega\}_{[8]}$
I	$\#\{CX, SWAP\}_{[2]}$
C^2T	$\#\{T, \omega, \omega\}_{[4]}$
C^3T	$\arg\det_3$