

Complete Equations for Qudit Circuits

via polycontrolled PROPs

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Qudits have d named states

Definition

A qudit wire represents a d -dimensional space with computational basis

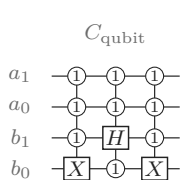
$$|0\rangle, |1\rangle, \dots, |d-1\rangle.$$

An n -wire circuit represents a unitary on $(\mathbb{C}^d)^{\otimes n}$.

Qudit platforms

- ▶ ion levels: trapped-ion qudits ($d \leq 7$)
[Ringbauer 22] [Hrmo 23]
- ▶ photonic modes: integrated ququart processors [Chi 22]
- ▶ transmon levels: qutrit/ququart control
[IBM] [Fischer 23]
- ▶ bosonic modes: cat-code systems, related but not qudits [Alice & Bob]

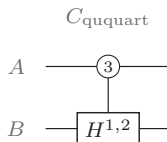
More structure per wire



basis
encoding

$$A = 2a_1 + a_0$$
$$B = 2b_1 + b_0$$

→
same semantics
after encoding



Compression example

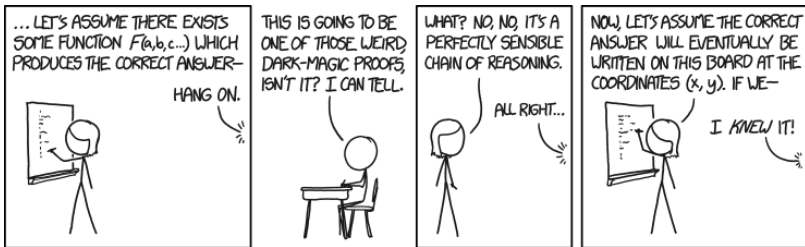
Packing two qubits into one ququart turns the branch 11 into the value $A = 3$, and the target pair 01, 10 into ququart levels 1, 2.

Circuit-level use cases

wire compression
temporary levels

pack binary data into higher-dimensional wires [Litteken 23]
store intermediate values during Toffoli decompositions [Gokhale 19]
[Nikolaeva 23]

Can every **true equality** between **exact**
qudit circuits be **proved**
using **finitely many local rewrite rules**?



Two-level gates give universality

Embedding a 2×2 unitary

$$U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}.$$

$U^{(i,i+1)}$ is the identity except on adjacent rows and columns $i, i+1$:

$$U^{(i,i+1)} = \begin{bmatrix} I & & & \\ & u_{00} & u_{01} & \\ & u_{10} & u_{11} & \\ & & & I \end{bmatrix}$$

rows and columns ordered as $\dots, i, i+1, \dots$

One-qudit synthesis

On the active two-level subspace:

$$P_0(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix},$$

$$P_1(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

Euler angles write any $U(2)$ operation as a product of

$$P_0(\alpha), P_1(\beta), H.$$

Adjacent two-level embeddings generate $U(d)$. [Nielsen–Chuang] [Clements 16]

One entangling gate gives universality

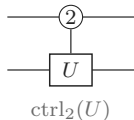
Entangling control

A value-control is a direct-sum branch.

For $d = 4$,

$$\text{ctrl}_2(U) = I \oplus I \oplus U \oplus I.$$

active branch $k = 2$



control-basis blocks

0	I			
1		I		
2			U	
3				I

$\text{diag}(I, I, U, I)$

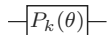
Universal gate set $\mathcal{G}_d$ [Brylinski–Brylinski 02]

$$\mathcal{L}_d = \{P_k(\theta), H^{(i,j)}\},$$

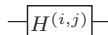
$$\mathcal{G}_d = \mathcal{L}_d \cup \{\text{ctrl}_1(P_1(\theta))\},$$

$$\langle \mathcal{G}_d \rangle = U(d^n).$$

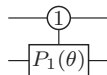
one-qudit



one-qudit



entangling



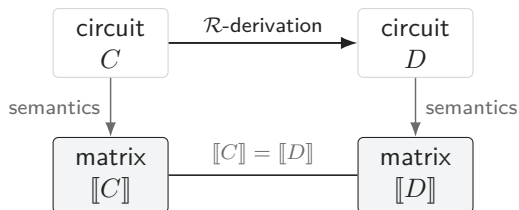
What is a complete presentation?

Definition

A presentation is a gate signature Σ and equations $\mathcal{R}$, closed under circuit contexts.

$$\mathcal{R} \vdash C = D$$

means local applications of $\mathcal{R}$ rewrite C into D .



Soundness: derivations preserve semantics.

Completeness: equal semantics give a derivation.

Example

Two adjacent H gates cancel; the same local rewrite is allowed inside any circuit context.

$$\text{---} \boxed{H} \boxed{H} \text{---} \xrightarrow{\mathcal{R}} \text{---}$$

Compression

Basis encoding

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \stackrel{\text{enc}}{\cong} \mathbb{C}^4, \quad |a_1 a_0\rangle \mapsto |2a_1 + a_0\rangle$$

Compression of 4 basis states

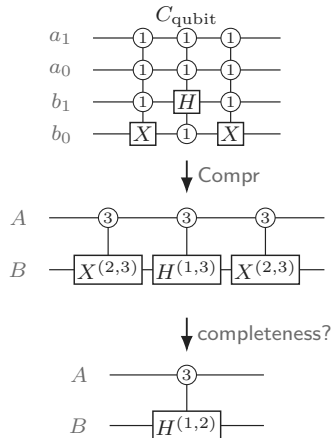
$$\text{Compr}(D \circ C) = \text{Compr}(D) \circ \text{Compr}(C),$$

$$\text{Compr}(\text{ctrl}_1(\text{ctrl}_1(U))) = \text{ctrl}_3(\text{Compr}(U)),$$

$$\text{Compr}(\text{ctrl}_1(X)) = X^{(2,3)},$$

$$\text{Compr}(\sigma \circ \text{ctrl}_1(H) \circ \sigma) = H^{(1,3)}.$$

△ This function here is partially defined.

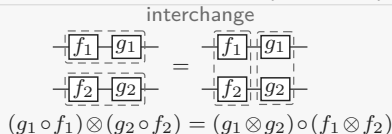
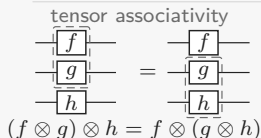
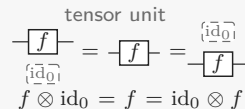
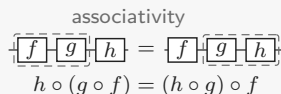
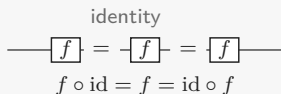
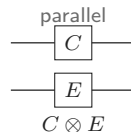
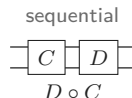


A PRO formalizes composition

Definition

A **PRO** is a strict monoidal category $\mathcal{P}$ with objects $\mathbb{N}$, tensor $m \otimes n = m + n$, and unit 0.

Read n as n **wires**; a morphism $C : n \rightarrow m$ as a circuit with n inputs and m outputs.



A PROP makes permutations structural

Definition

A **PROP** is a PRO with symmetries

$$\sigma_{m,n} : m + n \rightarrow n + m.$$

Diagrammatically, wires may be permuted as structural language.

involution

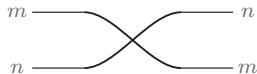


$\sigma_{n,m} \circ \sigma_{m,n} = \text{id}_{m+n}$

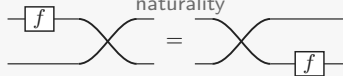
These are extra **symmetry laws**, on top of the PRO laws from the previous slide.

Unit: $\sigma_{0,n} = \text{id}_n = \sigma_{n,0}$.

$$\sigma_{m,n} : m + n \rightarrow n + m$$



naturality



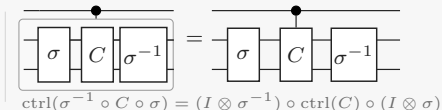
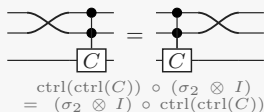
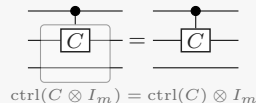
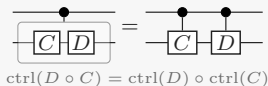
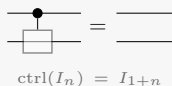
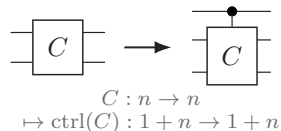
$\sigma \circ (f \otimes \text{id}) = (\text{id} \otimes f) \circ \sigma$

A controlled PROP adds control

Definition

A **control functor** is an operation $\text{ctrl} : \mathcal{P}_{\text{endo}} \rightarrow \mathcal{P}_{\text{endo}}$ with $\text{ctrl}(n) = 1 + n$: it adds one chosen control wire.

* $\mathcal{P}_{\text{endo}}$: all circuits $n \rightarrow n$.



One constructor per value

Definition

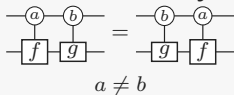
A d -ary **polycontrolled PROP** is a PROP equipped with control functors

$$(\text{ctrl}_k : \mathcal{P}_{\text{endo}} \rightarrow \mathcal{P}_{\text{endo}})_{k \in [d]}.$$



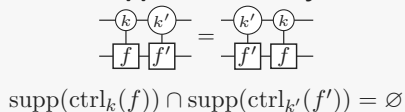
Four value-control principles

Commutativity

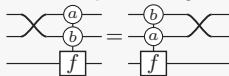


$\leq$

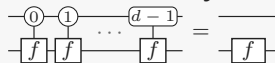
Support sensitivity



Compatibility



Exhaustivity



Several ways to prove completeness

Normal forms

$$C \xLeftrightarrow[\text{unique}] N(U) \xLeftrightarrow D$$

- + canonical representative
- + can become an algorithm
- hard to design for rich syntax
- must prove actual normalization

[Selinger 15] [Li et al. 26]

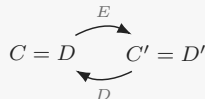
Group presentation

$$\langle G \mid R \rangle \cong \text{Im}[\![-]\!]$$

- + proves generators and relations
- + good for discrete fragments
- relation sets can explode
- less natural with real parameters

[Bian–Selinger 22] & [23]

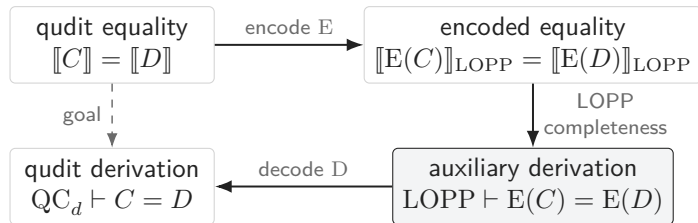
Transfer



- + reuses a complete calculus
- + isolates E, D , and mimicry
- expressivity must match
- auxiliary rules must decode

[Clément et al. 22]

Completeness is transferred by translation



Translation checks

E preserves semantics, $D(E(C)) = C$ is derivable, and each LOPP proof step must mimic inside QC_d .

$\triangle$ LOPP uses $\oplus$, not $\otimes$.
completeness: [Clément et al. 22]
simplified: [Heurtel 25]

Equational theory

$$(\text{Sum}) \quad \alpha \quad \beta = \alpha + \beta$$

$$(2\pi) \quad \textcircled{2\pi} = \boxed{}$$

$$(H^2) \quad \text{---} H^{(i,i+1)} \text{---} H^{(i,i+1)} \text{---} = \text{---}$$

$$(\text{XH}) \quad \text{---} X^{(i,j)} \text{---} H^{(j,k)} \text{---} = \text{---} H^{(i,k)} \text{---} X^{(i,j)} \text{---}$$

$$(\text{EH}) \quad \text{---} H^{(i,j)} \text{---} \textcircled{j}_{\alpha_0} \text{---} H^{(i,j)} \text{---} \textcircled{j}_{\alpha_2} \text{---} H^{(i,j)} \text{---} = \text{---} \textcircled{j}_{\beta_0} \text{---} H^{(i,j)} \text{---} \textcircled{i}_{\beta_1} \text{---} \textcircled{j}_{\beta_2} \text{---} H^{(i,j)} \text{---} \textcircled{j}_{\beta_3} \text{---}$$

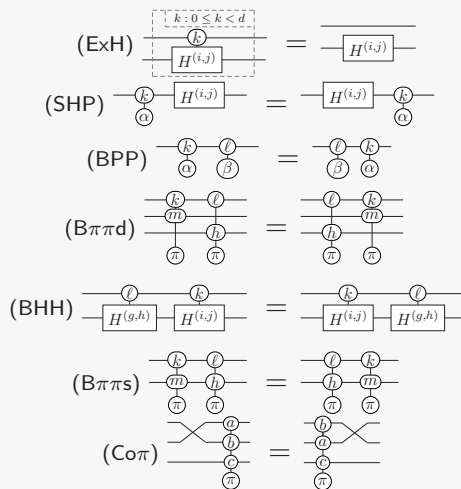
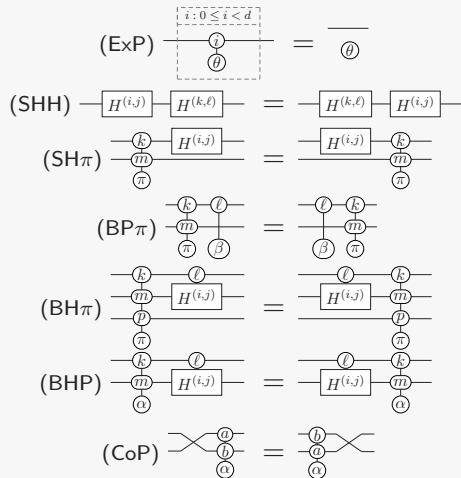
$$(3\text{Rx}) \quad \text{---} R_X^{(b,d)}(\gamma_1) \text{---} R_X^{(d,c)}(\gamma_2) \text{---} R_X^{(b,d)}(\gamma_3) \text{---} = \text{---} R_X^{(d,c)}(\delta_1) \text{---} R_X^{(b,d)}(\delta_2) \text{---} R_X^{(d,c)}(\delta_3) \text{---}$$

$$(\text{CXC}) \quad \text{---} \textcircled{c} \text{---} X^{(a,b)} \text{---} \textcircled{b} \text{---} X^{(c,d)} \text{---} \textcircled{c} \text{---} = \text{---} \textcircled{a} \text{---} X^{(c,d)} \text{---} \textcircled{d} \text{---} X^{(a,b)} \text{---} \textcircled{a} \text{---}$$

$$(\text{S}) \quad \text{---} \textcircled{a} \text{---} X^{(a,b)} \text{---} \textcircled{a} \text{---} X^{(a,b)} \text{---} \textcircled{a} \text{---} X^{(a,b)} \text{---} = \text{---} \text{---}$$

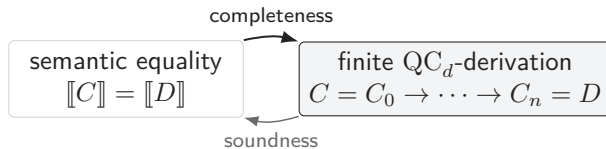
(a, b) : 0 ≤ a < b < d

Recovering the properties



Every equality has a QC_d -derivation

Theorem. For every finite dimension $d \geq 2$, the finite theory QC_d is **sound and complete** for exact unitary qudit circuits, including global phase.



exact: unitary equality, not approximation or noise

finite: fixed finite schemata for each d

local: bounded circuit windows inside contexts

Implications

- ▶ certificates for exact qudit rewrites
- ▶ explicit branch syntax
- ▶ gate equations separate from control algebra

Future work

- ▶ orient fragments as rewrite systems
- ▶ native and fault-tolerant gate sets
- ▶ arithmetic controls; fragment comparison