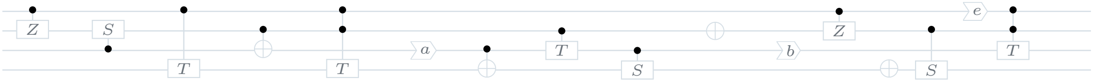

Completeness for prime-dimensional phase-affine circuits

Normal forms and complete rewrites for binomial phases

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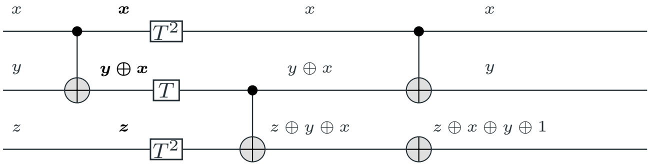
CNOT-dihedral is Hadamard-free Clifford+T

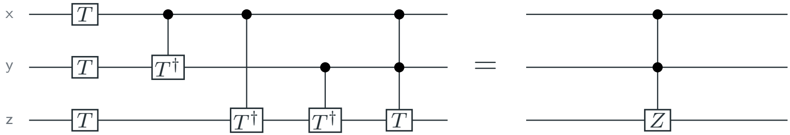
Definition

For $C \in \langle X, \mathrm{CNOT}, T\rangle$, $[\![C]\!]|x\rangle = \omega^{q(x)}|g(x)\rangle$,
where g is affine, $q : \mathbb{F}_2^n \to \mathbb{Z}_8$ has degree ≤ 3, and $\omega = e^{2\pi i/8}$.



$$q(x, y, z) = (y \oplus x) + 2z + 2x = (y + x - 2xy) + 2z + 2x$$
$$|x, y, z\rangle \mapsto \omega^{q(x,y,z)}|x, y, z \oplus x \oplus y \oplus 1\rangle$$

Spider nests encode CCZ parity



$$x + y + z - (x \oplus y) - (x \oplus z) - (y \oplus z) + (x \oplus y \oplus z) \equiv 4xyz \pmod{8}.$$

Proposition

The left-hand phase-gadget product has exponent $4xyz \pmod{8}$, hence it is exactly the controlled-controlled-Z phase on three Boolean wires.

[Amy–Chen–Ross, 2018] [de Beaudrap et al., 2020] [Kissinger–van de Wetering, 2024]

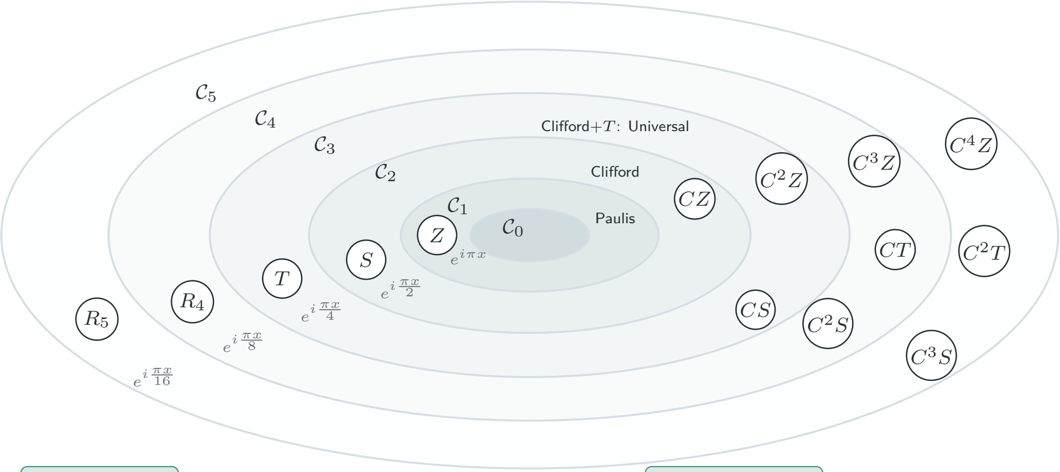
CNOT-dihedral normal forms compare coefficients

$$\begin{tikzcd} C \arrow[r, bend left, "\text{normalise}"] & \begin{matrix}\mathrm{NF}(C) = \mathrm{NF}(C') \\ D_p\, P_g\end{matrix} \arrow[l, bend left, "\text{expand}"] \arrow[r, bend right, "\text{expand}"'] & C' \arrow[l, bend right, "\text{normalise}"'] \end{tikzcd}$$

Theorem

Every $C \in \langle X, \mathrm{CNOT}, T \rangle$ rewrites to a unique phase-affine normal form $D_q P_g$.
Hence $\mathcal{E} \vdash C = C'$ iff $g = g'$ and $q = q'$.

- Clifford+T T-depth optimisation
  [Amy–Maslov–Mosca, 2014]
- non-Clifford randomized benchmarking
  [Cross et al., 2016]
- finite CNOT-dihedral presentations
  [Amy–Chen–Ross, 2018]
- CNOT-phase CNOT-cost synthesis
  [Amy–Azimzadeh–Mosca, 2018]
- optimal two-qubit CNOT-dihedral synthesis
  [Garion et al., 2020]
- spider-nest T-count reduction
  [de Beaudrap et al., 2020]
- scalable ZX spider nests
  [Kissinger–van de Wetering, 2024]
- minimal CNOT-dihedral presentation
  [Blake, 2026]



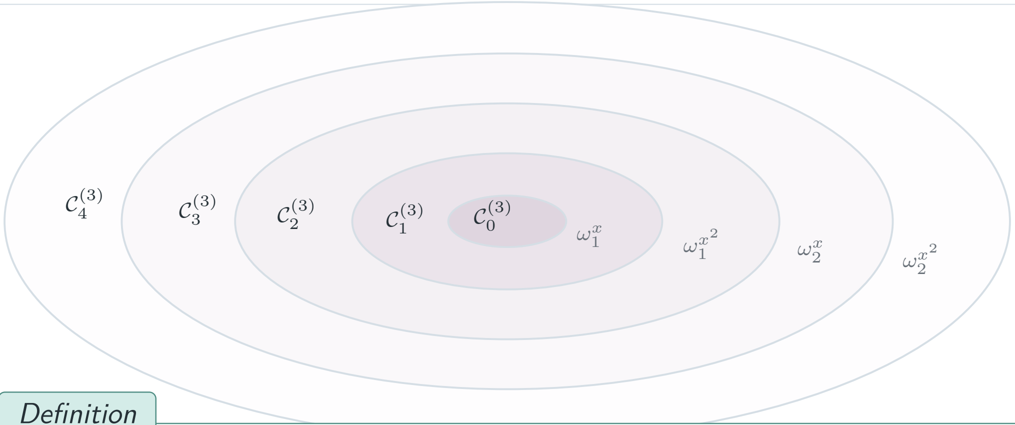
Definition

$$\mathcal{C}_{k+1} = \{U \mid UPU^\dagger \in \mathcal{C}_k,\ \forall P \in \text{Pauli}\}.$$

Proposition

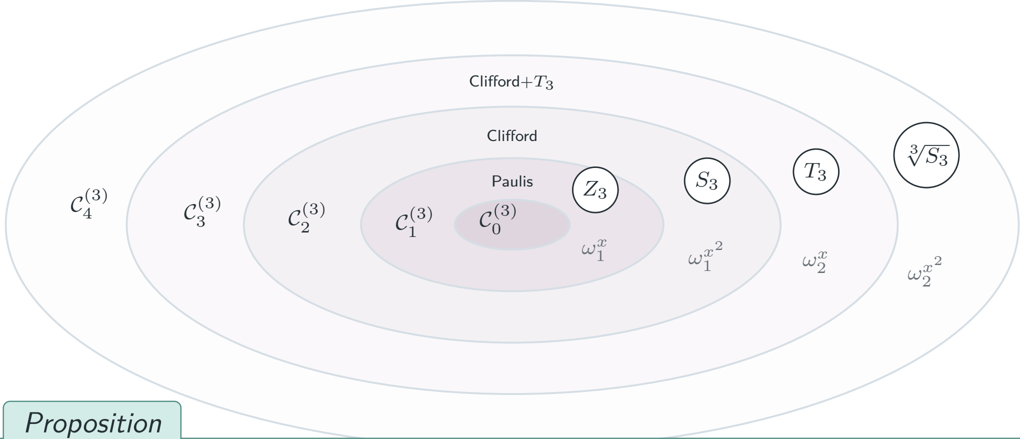
$$D \in \mathcal{C}_k \implies \text{Controlled } D \in \mathcal{C}_{k+1}.$$

Qutrit hierarchy



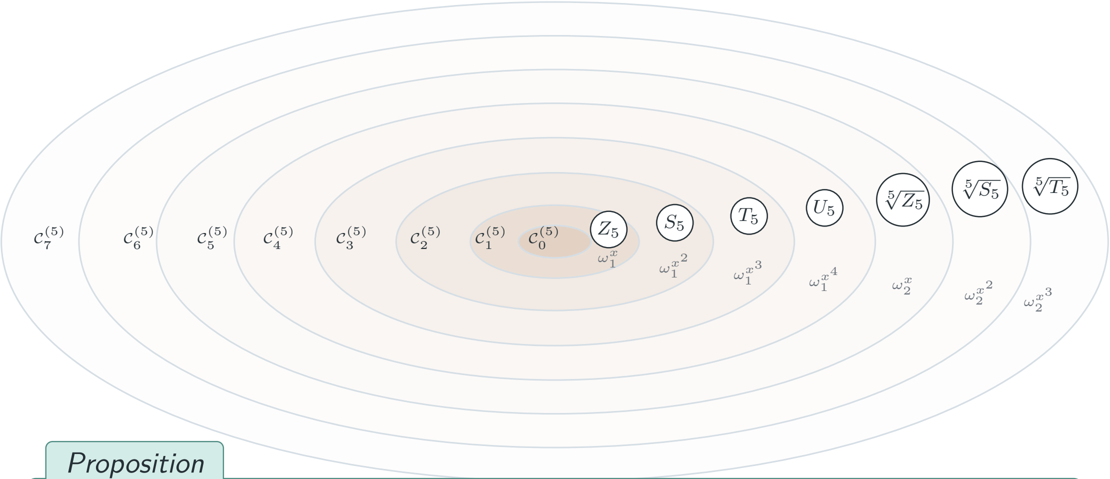
Definition

For non-scalar one-qutrit diagonal hierarchy gates $D_{r,f}|x\rangle = \omega_r^{f(x)}|x\rangle$, $\omega_r = e^{2\pi i/3^r}$, the level is $2(r-1) + \deg f$ with $\deg f \in \{1, 2\}$. [Cui–Gottesman–Krishna, 2017]



Proposition

- $T_3 = \mathrm{diag}(1, \omega_2, \omega_2^8)$, with $\omega_2 = e^{2\pi i/9}$.
- $T_3 \in \mathcal{C}_3^{(3)} \setminus \mathcal{C}_2^{(3)}$ and $T_3^3 = Z_3$.
- Clifford+T_3 is approximately universal [Campbell–Anwar–Browne, 2012]; [Glaudell et al., 2022].



Proposition

- In dimension 5, $T_5|x\rangle = \omega_1^{x^3}|x\rangle$ lies in $\mathcal{C}_3^{(5)} \setminus \mathcal{C}_2^{(5)}$ [Cui–Gottesman–Krishna, 2017].
- Fifth roots of Z_5, S_5, T_5 occupy different higher levels.
- We use the third-level non-Clifford phase as the T-gate convention.

Qutrit T uses finite differences

$[\![S]\!] = \omega^{x^2}$	$[\![S]\!] = \omega^{\binom{x}{2}} = \omega^{\frac{x(x-1)}{2}}$
$SXS^\dagger = \omega^{2x+1}X$	$SXS^\dagger = \omega^{x}X$
Result $\omega Z^2 X$	Result ZX

Proposition

The usual qutrit phase gate is the diagonal operator $S_3|x\rangle = \omega^{\frac{x(x-1)}{2}}|x\rangle = \omega^{\binom{x}{2}}|x\rangle$, because $\Delta\binom{x}{2} = \binom{x+1}{2} - \binom{x}{2} = x$.

Binomial identities generate diagonal gates

$$q(\mathbf{x}) = \sum_i A_i \mathbf{x}_i + \sum_i B_i \mathbf{x}_i^2 + \sum_i T_i \mathbf{x}_i^3 + \sum_{i<j} C_{ij} \mathbf{x}_i \mathbf{x}_j$$

$$+ \sum_{i<j} U_{ij} \mathbf{x}_i \mathbf{x}_j^2 + \sum_{i<j} \bar{U}_{ij} \mathbf{x}_i^2 \mathbf{x}_j + \sum_{i<j<k} V_{ijk} \mathbf{x}_i \mathbf{x}_j \mathbf{x}_k .$$

monomial basis

Stirling transform

triangular inverse

$$q(\mathbf{x}) = \sum_i a_i \mathbf{x}_i + \sum_i b_i \binom{\mathbf{x}_i}{2} + \sum_i t_i \binom{\mathbf{x}_i}{3} + \sum_{i<j} c_{ij} \mathbf{x}_i \mathbf{x}_j$$

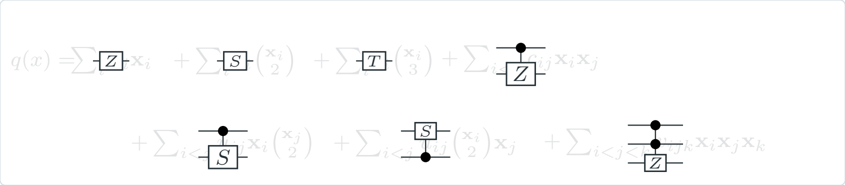
$$+ \sum_{i<j} u_{ij} \mathbf{x}_i \binom{\mathbf{x}_j}{2} + \sum_{i<j} \bar{u}_{ij} \binom{\mathbf{x}_i}{2} \mathbf{x}_j + \sum_{i<j<k} v_{ijk} \mathbf{x}_i \mathbf{x}_j \mathbf{x}_k .$$

binomial basis

Proposition

For degree $< d$, the monomial and binomial bases are related by an invertible triangular change of basis.

Binomial identities generate diagonal gates



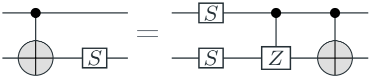
Proposition

For degree $< d$, the monomial and binomial bases are related by an invertible triangular change of basis. The ordered diagonal layer records the same polynomial as a binomial coefficient vector.

Affine transport is binomial

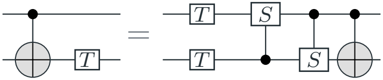
With binomials

$$[\![S]\!]_{\text{bin}} = \omega^{\binom{x}{2}}$$



$$\binom{x+y}{2} = \binom{x}{2} + \binom{y}{2} + xy$$

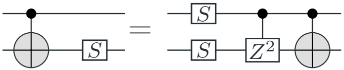
$$[\![T]\!]_{\text{bin}} = \omega^{\binom{x}{3}}$$



$$\binom{x+y}{3} = \binom{x}{3} + \binom{y}{3} + y\binom{x}{2} + x\binom{y}{2}$$

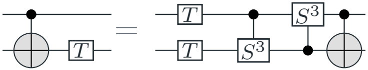
With monomials

$$[\![S]\!]_{\text{mon}} = \omega^{x^2}$$



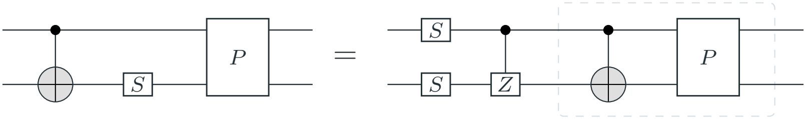
$$(x+y)^2 = x^2 + y^2 + 2xy$$

$$[\![T]\!]_{\text{mon}} = \omega^{x^3}$$



$$(x+y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$$

Normalisation collects phases diagonally



Proposition

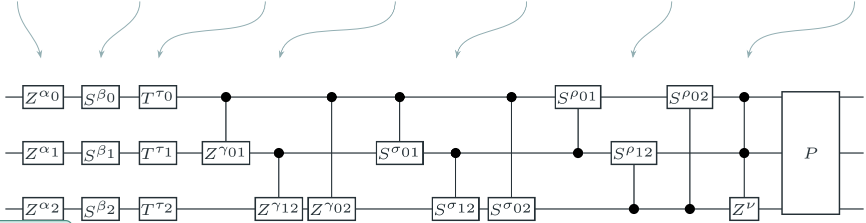
Let A_g be an affine generator and D_q a binomial diagonal generator. Then $A_g D_q = D_{q\circ g} A_g$, with $q \circ g$ expanded in the ordered binomial basis.

Theorem

Every phase-affine circuit C rewrites as $\mathcal{E} \vdash C = D_q P_g$, where P_g is the affine normal form and q is a binomial coefficient vector.

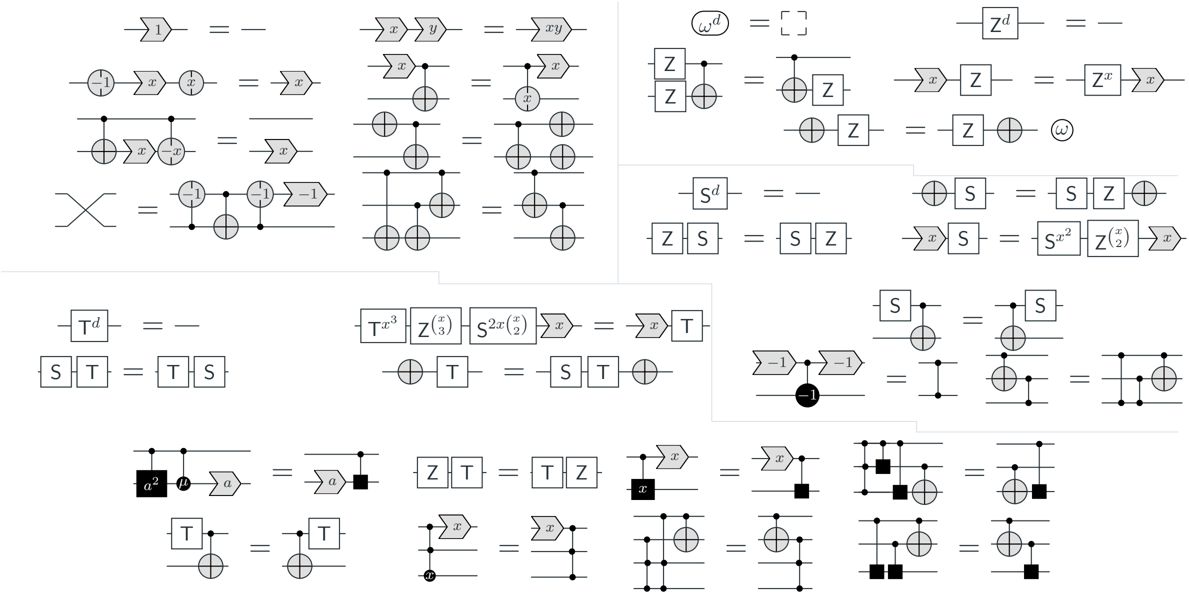
Diagonals share one binomial form

$$q(x) = \sum_i \alpha_i \mathbf{x}_i + \sum_i \beta_i \binom{\mathbf{x}_i}{2} + \sum_i \tau_i \binom{\mathbf{x}_i}{3} + \sum_{i<j} \gamma_{ij} \mathbf{x}_i \mathbf{x}_j + \sum_{i<j} \sigma_{ij} \mathbf{x}_i \binom{\mathbf{x}_j}{2} + \sum_{i<j} \rho_{ij} \binom{\mathbf{x}_i}{2} \mathbf{x}_j + \sum_{i<j<k} \nu_{ijk} \mathbf{x}_i \mathbf{x}_j \mathbf{x}_k.$$



Theorem

For n wires and degree ≤ 3, every circuit rewrites to $D_q P_g$, with P_g the affine normal form and q the ordered binomial coefficient vector. The pair (g, q) is unique.



Completeness follows from normal-form uniqueness

Theorem

For circuits C, C' in $\mathbf{LinPhase}_d$ (d prime), $\mathbf{QuadPhase}_d$ (d odd), or $\mathbf{CubicPhase}_d$ ($d > 3$),

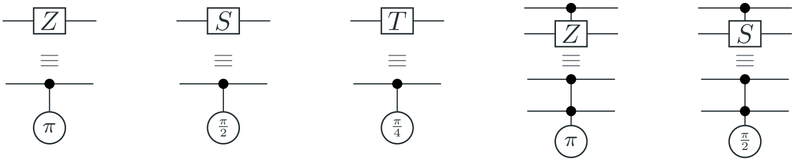
$$[\![C]\!]_{\mathcal{P}} = [\![C']\!]_{\mathcal{P}} \iff \mathcal{E}_{\mathcal{P}} \vdash C = C'.$$

$$\begin{tikzcd}
C \arrow[r, "{[\![\cdot]\!]_{\mathcal{P}}}"] \arrow[d, "\mathrm{NF}"'] & {[\![C]\!]_{\mathcal{P}} = [\![C']\!]_{\mathcal{P}}} \arrow[d] & C' \arrow[l, "{[\![\cdot]\!]_{\mathcal{P}}}"'] \arrow[d, "\mathrm{NF}"] \\
\mathrm{NF}(C) = D_q P_g \arrow[dr] & \text{uniqueness of } (g, q) \arrow[d] & \mathrm{NF}(C') = D_{q'} P_{g'} \arrow[dl] \\
& g = g',\ q = q' &
\end{tikzcd}$$

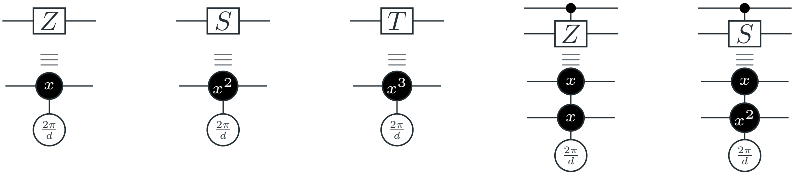
$\mathrm{NF}(C) = \mathrm{NF}(C')$, hence $\mathcal{E}_{\mathcal{P}} \vdash C = C'$.

Modern phase notations

On qubits



On qudits



Conclusion

Takeaway.

- CNOT-dihedral: affine Boolean map plus $\mathbb{Z}_8$-valued cubic phase polynomial.
- Prime qudits: affine $\mathbb{F}_d$-map plus finite-field binomial phase polynomial.
- Completeness: transport, collect, reduce finite orders, compare (g, q).

Open directions.

- Minimal and reduced finite presentations.
- Optimisation passes for qudit phase-affine circuits.
- ZX- and ZH-style presentations of binomial/spider-nest identities.
- Higher Clifford-hierarchy fragments.

Proposition

The Clifford hierarchy is classified by phase precision and polynomial degree. Binomial coordinates are the coordinate system in which affine transport is triangular.