

Simpler Presentations for Many Fragments of Quantum Circuits

Colin Blake 

Inria Mocqua and Université de Lorraine, CNRS, LORIA, F-54000 Nancy, France

Abstract

Equational reasoning is central to quantum circuit optimisation and verification: one replaces subcircuits by provably equivalent ones using a fixed set of rewrite rules viewed as equations. A finite rule set is most informative when it separates the genuine algebra of a circuit fragment from the structural treatment of wires. This paper gives six near-Clifford fragments a common PROP treatment, where wire permutations are structural: qubit Clifford, real Clifford, Clifford+T (up to two qubits), Clifford+CS (up to three qubits), CNOT-dihedral, and qutrit Clifford. Starting from prior completeness theorems, we transfer completeness into this setting and remove redundant non-structural rules, then check minimality by separating interpretations tailored to individual axioms; the resulting presentations are minimal in all arities for qubit Clifford, real Clifford, and CNOT-dihedral, minimal in bounded ranges for the remaining fragments, and comparable by one transfer-and-separation pattern.

2012 ACM Subject Classification Theory of computation → Quantum computation theory

Keywords and phrases Quantum circuits, Clifford group, equational theories, minimality, qutrit

Digital Object Identifier 10.4230/LIPIcs.FSCD.2026.3

Funding This work is supported by the Plan France 2030 through the PEPR integrated project EPiQ (ANR-22-PETQ-0007) and the HQI platform (ANR-22-PNCQ-0002); by the European Union through the MSCA Staff Exchanges project QCOMICAL (Grant Agreement ID: 101182520); and by the Maison du Quantique MaQuEst.

Acknowledgements I thank my PhD advisors Simon Perdrix and Miriam Backens for reviews of early versions of the article. I also thank Noé Delorme for many discussions, early presentations of the Clifford and Real Clifford fragments, and joint attempts to derive candidate rules.

1 Introduction

Optimisation and verification of quantum circuits are often carried out by *equational reasoning*: one rewrites a circuit by replacing a subcircuit with a provably equivalent one using a fixed set of identities [?, ?]. For a fixed gate signature, this is formalised as an equational theory, namely a set of well-typed equations between circuits, closed under sequential and parallel composition [?, ?, ?]. With respect to an intended interpretation by unitary operators, a rule set is *complete* if it derives every equality between circuits having the same denotation.

A finite complete presentation then gives a compact algebraic description of a circuit fragment: it records which identities are taken as primitive, which ones are consequences, and how that distinction depends on the ambient syntax. If wire permutations are represented by ordinary gates, then naturality and swap-generator interaction equations describe how those permutations interact with the generators of the fragment; if permutations are structural, those equations become part of the surrounding calculus. Reducing a presentation is therefore more than shortening a list of rules, because it can reveal which equalities belong to generic wiring and which belong to the algebra of the fragment itself.

Alongside completeness, we study *axiom independence* and *minimality* [?, ?, ?]: an axiom is independent if it is not derivable from the others, and a finite presentation is minimal if all its axioms are independent. Minimal presentations isolate the irreducible algebraic content



© Colin Blake;

licensed under Creative Commons License CC-BY 4.0

11th International Conference on Formal Structures for Computation and Deduction (FSCD 2026).

Editor: Frank Pfenning; Article No. 3; pp. 3:1–3:75

Leibniz International Proceedings in Informatics



LIPIC Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

of a fragment and are also attractive for automated rewriting, where redundant rules enlarge the search space [?, ?]. For the fragments studied here, minimality also gives a useful check on the simplification process: after derivable rules have been removed, each remaining axiom is tested against the possibility that it was already forced by the others.

This work studies six near-Clifford fragments—qubit Clifford, real Clifford, Clifford+ T up to two qubits, Clifford+ CS up to three qubits, CNOT-dihedral, and qutrit Clifford—and places them in a common PROP setting, so that swaps are structural rather than fragment-specific. In this setting, objects are wire counts, morphisms are circuits, sequential composition is plugging, tensor is side-by-side juxtaposition, and the PROP structure accounts for wire crossings, thereby separating generic wiring from the fragment-specific equations whose size and independence we want to analyse.

| Fragment | Dimension | Non-structural generators |
|-----------------|-----------|---|
| Qubit Clifford | 2 | $\omega_8 : 0 \rightarrow 0$, H , S : $1 \rightarrow 1$, CNOT : $2 \rightarrow 2$ |
| Real Clifford | 2 | $\omega_8 : 0 \rightarrow 0$, H , Z : $1 \rightarrow 1$, CNOT : $2 \rightarrow 2$ |
| Clifford+ T | 2 | $\omega_8 : 0 \rightarrow 0$, H , T : $1 \rightarrow 1$, CNOT : $2 \rightarrow 2$ |
| Clifford+ CS | 2 | $\omega_8 : 0 \rightarrow 0$, H , S : $1 \rightarrow 1$, CNOT : $2 \rightarrow 2$ |
| CNOT-dihedral | 2 | $\omega_8 : 0 \rightarrow 0$, X , T : $1 \rightarrow 1$, CNOT : $2 \rightarrow 2$ |
| Qutrit Clifford | 3 | $\omega_{12} : 0 \rightarrow 0$, H , S : $1 \rightarrow 1$, CNOT : $2 \rightarrow 2$ |

■ **Table 1** Gate signatures used for the six presented fragments; swaps are structural and therefore not listed as fragment generators.

The source presentations in the literature are therefore our point of departure: after moving to a common PROP setting with structural symmetries, we seek presentations with fewer non-structural axioms and separator data for the minimality bounds listed in Table 2, so simplification, completeness transfer, and minimality become one comparison rather than separate rule-count exercises.

1.1 Related work

Generators-and-relations completeness theorems already give presentations for all six fragments of Table 1: the n -qubit Clifford presentation [?], the real Clifford presentation [?], the CNOT-dihedral presentation [?], the 2-qubit Clifford+ T presentation [?], the 3-qubit Clifford+ CS presentation [?], and the n -qutrit Clifford presentation [?]. These theorems are the starting point for the completeness transfer in Section 4: explicit encodings and decodings carry the known complete presentations to the smaller PROP presentations of Section 3. Related Agda formalizations already check the 2-qubit Clifford+ T and 3-qubit Clifford+ CS presentations [?, ?].

Beyond the near-Clifford setting, recent completeness theorems cover more expressive circuit languages [?, ?], where the extra expressivity changes the shape of the axiomatisation: for such languages, complete equational theories require rules of unbounded arity [?]. The latter work also uses separating models to prove minimality for a complete theory of general quantum circuits, and Section 5 adapts that proof pattern to the six fragments of Table 1. This work preserves the strict unitary semantics of the earlier completeness papers and asks which axioms remain necessary after simplification.

1.2 Contributions

1. **Smaller presentations in a uniform PROP setting.** For each fragment we give a finite presentation with fewer non-structural axioms than the standard one while preserving the same strict unitary semantics.
2. **Completeness by transfer.** The transfer isolates a finite syntactic comparison: explicit encoding/decoding maps carry earlier complete presentations to the smaller ones after aligning the ambient PROP structure and scalar conventions where necessary.
3. **Minimality and independence results for the simplified rule sets.** By separating interpretations, we prove that the new presentations for qubit Clifford, real Clifford, and CNOT-dihedral are minimal in all arities, while the Clifford+ T , Clifford+ CS , and qutrit Clifford presentations are minimal in the bounded ranges listed in Table 2.

The arity bounds have two origins: for Clifford+ T and Clifford+ CS , the imported completeness theorems themselves are proved only up to 2 and 3 qubits, while for minimality the remaining bounds are the arities for which we construct explicit separating interpretations.

After the generic PROP structure has been factored out, Table 2 compares non-structural rule counts; the removed source rules are not lost, since they are derivable from the simplified presentations.

| Fragment | Previous Presentation | Our Presentation | Figure |
|--------------------------------|-----------------------|--|--------|
| Qubit Clifford on n qubits | 15 rules [?] | 8 rules (minimal in all arities) | 2 |
| Real Clifford on n qubits | 16 rules [?] | 10 rules (minimal in all arities) | 3 |
| Qutrit Clifford on n qutrits | 18 rules [?] | 10 rules (minimal up to 2 qutrits; conjectured in all arities) | 4 |
| Clifford+ T on 2 qubits | 18 rules [?] | 11 rules (minimal up to 1 qubit) | 5 |
| Clifford+ CS on 3 qubits | 17 rules [?] | 14 rules (minimal up to 2 qubits) | 6 |
| CNOT-dihedral on n qubits | 13 rules [?] | 11 rules (minimal in all arities) | 7 |

■ **Table 2** Non-structural rule-count comparison after moving from source PRO presentations to PROP presentations with structural symmetries.

The rest of the paper follows the proof obligations behind Table 2: Section 2 fixes the PROP syntax used to lift PRO presentations, Section 3 defines the six presented circuit fragments and their unitary interpretations, Section 4 transfers completeness from the source presentations and handles scalar refinements, Section 5 checks independence by separating interpretations, and the appendices collect the longer derivations and case-by-case checks supporting the main proofs.

2 Graphical languages for quantum circuits

The formal language is categorical, but its role is practical: it fixes the syntax for the usual boxes-and-wires calculus. In this setting, generic wiring is treated structurally, so the displayed rule sets can focus on fragment-specific generators and equations.

2.1 Graphical languages

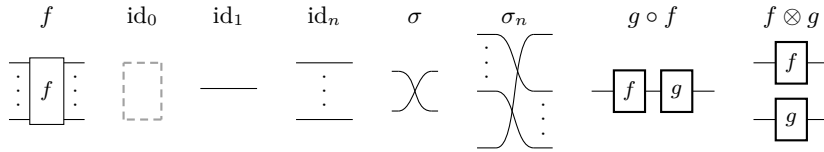
We work with the one-object-per-wire formalisms PROs and PROPs in the sense of Joyal–Street [?, ?]. We adopt strictness throughout: associativity and unit constraints are treated as literal equalities, so parentheses can be omitted.

► **Definition 1.** A PRO is a strict monoidal category \mathbf{P} whose objects are the natural numbers $n \in \mathbb{N}$ and whose tensor on objects is addition $n \otimes m := n + m$ with unit 0. We write $\mathbf{P}(n, m)$ for morphisms $n \rightarrow m$, with sequential composition \circ and parallel composition \otimes , subject to the usual strict monoidal axioms (Figures 1a–1e).

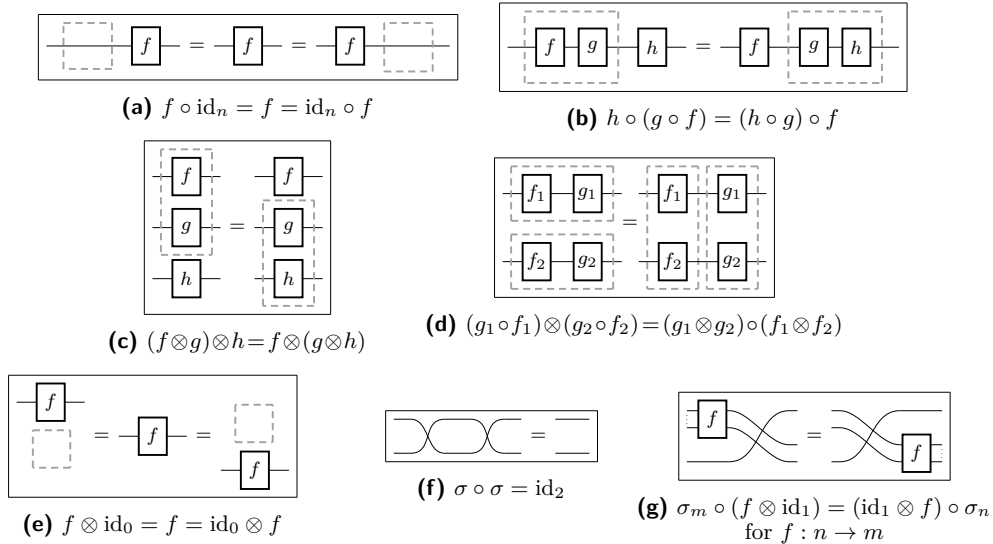
We write $f : n \rightarrow m$ when $f \in \mathbf{P}(n, m)$; n and m are the numbers of input and output wires of f . In particular $\text{id}_n = \text{id}_1^{\otimes n}$ for all $n \in \mathbb{N}$.

► **Definition 2.** A PROP is a PRO \mathbf{P} equipped with a symmetric monoidal structure, i.e. symmetry isomorphisms $\sigma_{n,m} : n + m \rightarrow m + n$ ($n, m \in \mathbb{N}$) generated by the basic swap $\sigma := \sigma_{1,1}$ and satisfying the axioms (1f)–(1g) in Figure 1. We write $\sigma_n := \sigma_{n,1} : n + 1 \rightarrow 1 + n$ for swapping one wire with a block of n wires.

In a PROP, swaps and permutations are part of the ambient structure, and the resulting string-diagram representation draws a morphism $f : n \rightarrow m$ as a box with n incoming wires and m outgoing wires. Sequential composition is drawn by stacking boxes, tensor by placing them side by side, and $\sigma_{n,m}$ by crossing a block of n wires with a block of m wires. With these conventions, the coherence axioms are exactly the diagrammatic moves that make deformation sound. For example, if $g : 1 \rightarrow 1$ and $h : 2 \rightarrow 2$ are gates, then $g \otimes h : 3 \rightarrow 3$ is their side-by-side circuit, and composing with $\sigma_{1,2}$ just moves the first output wire past the other two. Such crossings are part of the syntax before any fragment-specific rule is added.



Equalities obtained solely by these coherence moves, including permutations, are *structural* and may be omitted from diagrammatic arguments.



■ **Figure 1** Coherence laws for PROs and PROPs.

2.2 Free PROPs and presentations

Circuit fragments are specified by *generators and relations*. Generators represent primitive gates with fixed arity, and relations are the rewrite rules used for equational reasoning.

► **Definition 3.** A (PROP) signature is a set Σ equipped with arity functions $\text{dom}, \text{cod} : \Sigma \rightarrow \mathbb{N}$. An element $g \in \Sigma$ is written as a generating morphism $g : \text{dom}(g) \rightarrow \text{cod}(g)$.

► **Definition 4.** Let Σ be a signature. The free PROP on Σ , written \mathbf{P}_Σ , is the PROP whose morphisms are string diagrams generated from:

- the non-structural generators $g \in \Sigma$ (with their prescribed arities), and
 - the structural morphisms of a PROP (identities and symmetries),
- quotiented by PROP coherence (i.e. the axioms of Figure 1).

► **Definition 5.** Let Σ be a signature and let \mathcal{R} be a set of well-typed equations $L = R$ between morphisms of the free PROP \mathbf{P}_Σ . The presented PROP $\mathbf{P}_\Sigma/\mathcal{R}$ is the quotient of \mathbf{P}_Σ by the smallest congruence that contains \mathcal{R} and is closed under \circ , \otimes , and symmetry.

This quotient is the *graphical language* used in the rest of the paper: diagrams are equal exactly when their equality is derivable from PROP coherence together with the chosen relations.

2.3 Monoidal functors and completeness

An interpretation sends the presented syntax to its intended operations; completeness is the assertion that every semantic equality in that image can be recovered as a derivation in the quotient.

► **Definition 6.** A PROP morphism is a strict symmetric monoidal functor $F : \mathbf{P}_1 \rightarrow \mathbf{P}_2$ between PROPs that is the identity on objects. Equivalently, F preserves identities, symmetries, sequential composition and tensor.

► **Definition 7.** Let \mathbf{P} be a PROP. A subPROP of \mathbf{P} is a wide strict symmetric monoidal subcategory $\mathbf{Q} \subseteq \mathbf{P}$. Since the objects of a PROP are fixed to be \mathbb{N} , this amounts to specifying subsets $\mathbf{Q}(n, m) \subseteq \mathbf{P}(n, m)$ ($n, m \in \mathbb{N}$) such that $\text{id}_n \in \mathbf{Q}(n, n)$ for all n ; $\sigma_{n, m} \in \mathbf{Q}(n + m, m + n)$ for all n, m ; and \mathbf{Q} is closed under \circ and \otimes .

In this situation, the inclusion functor $\mathbf{Q} \hookrightarrow \mathbf{P}$ is a faithful PROP morphism.

► **Definition 8.** Let $\mathbf{P}_\Sigma/\mathcal{R}$ be a presented PROP and let \mathbf{C} be a symmetric monoidal category. An interpretation functor is a strict symmetric monoidal functor $\llbracket \cdot \rrbracket : \mathbf{P}_\Sigma \rightarrow \mathbf{C}$ specified on generators $g \in \Sigma$ and extended monoidally to all diagrams.

If every equation in \mathcal{R} holds in \mathbf{C} , then $\llbracket \cdot \rrbracket$ factors uniquely through the quotient, yielding a functor $\llbracket \cdot \rrbracket : \mathbf{P}_\Sigma/\mathcal{R} \rightarrow \mathbf{C}$.

► **Definition 9.** Let $\llbracket \cdot \rrbracket : \mathbf{P}_\Sigma/\mathcal{R} \rightarrow \mathbf{C}$ be such an interpretation. We say that the graphical language $\mathbf{P}_\Sigma/\mathcal{R}$ is complete for this semantics if, for all circuits C_1, C_2 , $\llbracket C_1 \rrbracket = \llbracket C_2 \rrbracket$ implies $\mathbf{P}_\Sigma/\mathcal{R} \vdash C_1 = C_2$. Equivalently, the interpretation functor is faithful.

We will also use a bounded variant. Given $k \in \mathbb{N}$, we say that $\mathbf{P}_\Sigma/\mathcal{R}$ is complete up to k wires if the implication holds whenever $C_1, C_2 : n \rightarrow m$ with $n, m \leq k$. Equivalently, the interpretation functor is faithful on each hom-set $\mathbf{P}_\Sigma/\mathcal{R}(n, m)$ for $n, m \leq k$.

For equational reasoning, completeness says that semantic equality can always be witnessed by an explicit diagrammatic derivation from the chosen relations; the bounded form restricts this requirement to circuits of limited arity.

When \mathbf{C} is, by definition, the symmetric monoidal subcategory of an ambient category generated by the images of the generators, the interpretation functor is also full; in that case $\mathbf{P}_\Sigma/\mathcal{R}$ is actually isomorphic to the semantic subPROP \mathbf{C} .

Completeness theorems stated for PROs can still be used in this PROP setting through a single lifting principle.

► **Lemma 10** (PRO-to-PROP lifting). *Let \mathbf{P}/\mathcal{R} be a complete presented PRO with an interpretation into a symmetric monoidal category \mathbf{C} . Assume there exists a PRO circuit $\tau : 2 \rightarrow 2$ such that $\llbracket \tau \rrbracket$ is the symmetry $\sigma_{1,1}$ of \mathbf{C} . Form the free PROP on the same generators, and quotient it by \mathcal{R} together with the single equation $\sigma_{1,1} = \tau$. Then the resulting PROP presentation is complete for the same semantics.*

Proof. The equation $\sigma_{1,1} = \tau$ rewrites each structural swap between adjacent wires as a source PRO circuit. By the symmetric monoidal axioms of a PROP, every structural permutation is generated by such adjacent swaps, so each PROP diagram D has a translated PRO diagram $T(D)$ obtained by replacing structural swaps by copies of τ and leaving the ordinary generators unchanged. The interpretation satisfies $\llbracket D \rrbracket = \llbracket T(D) \rrbracket$ because $\llbracket \tau \rrbracket = \sigma_{1,1}$. If two PROP diagrams have the same semantics, their translations have the same semantics as PRO diagrams, hence are equal by PRO completeness. Replacing each occurrence of τ by $\sigma_{1,1}$ then gives the original PROP equality. ◀

3 Quantum circuits and their relations

The transfer and separation arguments use one Hilbert-space semantic setup: five qubit fragments share a signature, while qutrit Clifford uses its own presented subPROP.

Let \mathbf{FdHilb} be the symmetric monoidal category of finite-dimensional complex Hilbert spaces and linear maps, with tensor product \otimes , tensor unit \mathbb{C} , and symmetry given by swapping tensor factors. By Mac Lane's coherence theorem [?] we may regard \mathbf{FdHilb} as strict monoidal.

► **Definition 11.** *Let H, K be finite-dimensional Hilbert spaces. A linear map $U : H \rightarrow K$ is unitary if $U^\dagger U = \text{id}_H$ and $UU^\dagger = \text{id}_K$, where U^\dagger denotes the Hilbert adjoint of U . We write $\mathbf{U}(H, K)$ for the set of unitary maps $H \rightarrow K$, and $\mathbf{U}(H) := \mathbf{U}(H, H)$.*

► **Definition 12.** *For $d \geq 2$, let \mathbf{Qudit}_d be the PROP with objects \mathbb{N} and hom-sets $\mathbf{Qudit}_d(n, m) := \mathbf{U}(\mathbb{C}^d{}^{\otimes n}, \mathbb{C}^d{}^{\otimes m})$, with composition, identities, tensor, and symmetry inherited from \mathbf{FdHilb} viewed as strict monoidal. The qubit and qutrit PROPs are $\mathbf{Qubit} := \mathbf{Qudit}_2$ and $\mathbf{Qutrit} := \mathbf{Qudit}_3$.*

For $d \geq 2$, \mathbf{Qudit}_d is endomorphism-only: there are no unitaries $n \rightarrow m$ for $n \neq m$. For a set G of morphisms in a PROP, $\langle G \rangle$ denotes the smallest subPROP containing G and all structural morphisms.

3.1 Qubit circuit fragments as presented subPROPs

The five qubit fragments share a common qubit signature and unitary interpretation.

► **Definition 13.** *The common qubit signature $\Sigma^{(2)}$ has non-structural generators $\otimes, \ominus : 0 \rightarrow 0$, $\overline{\text{H}}, \overline{\text{Z}}, \overline{\text{S}}, \overline{\text{T}}, \overline{\text{X}} : 1 \rightarrow 1$, and $\overline{\oplus}, \overline{\ominus} : 2 \rightarrow 2$. We write $\mathbf{P}^{(2)} := \mathbf{P}_{\Sigma^{(2)}}$ for the free PROP on $\Sigma^{(2)}$. Structural morphisms id_n and $\sigma_{n,m}$ come from the ambient PROP structure.*

► **Remark 14.** All signatures used in this section are endomorphism-only: each generator has type $n \rightarrow n$. Hence every well-typed fragment circuit, and every equation between such circuits, preserves the wire count; this is the arity fact used in Section 5.

We use the standard Dirac notation for computational basis states: for $x \in \{0, 1\}$ we write $|x\rangle \in \mathbb{C}^2$, and for $x, y \in \{0, 1\}$ we abbreviate $|x, y\rangle := |x\rangle \otimes |y\rangle$ (similarly, $|x_1, \dots, x_n\rangle := |x_1\rangle \otimes \dots \otimes |x_n\rangle$).

By the universal property of the free PROP, there is a unique strict symmetric monoidal functor $\llbracket \cdot \rrbracket^{(2)} : \mathbf{P}^{(2)} \rightarrow \mathbf{Qubit}$ sending:

$$\begin{array}{ll} \llbracket \omega_8 \rrbracket^{(2)} : 1 \mapsto e^{i\pi/4} & \llbracket \text{-H-} \rrbracket^{(2)} : |x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{k=0}^1 (-1)^{xk} |k\rangle \\ \llbracket \ominus \rrbracket^{(2)} : 1 \mapsto -1 & \llbracket \text{-S-} \rrbracket^{(2)} : |x\rangle \mapsto i^x |x\rangle \\ \llbracket \text{-X-} \rrbracket^{(2)} : |x\rangle \mapsto |1-x\rangle & \llbracket \text{-Z-} \rrbracket^{(2)} : |x\rangle \mapsto (-1)^x |x\rangle \\ \llbracket \text{-CNOT-} \rrbracket^{(2)} : |x, y\rangle \mapsto |x, x \oplus y\rangle & \llbracket \text{-T-} \rrbracket^{(2)} : |x\rangle \mapsto e^{i\pi x/4} |x\rangle \\ \llbracket \text{-CNOT-} \rrbracket^{(2)} : |x, y\rangle \mapsto i^{xy} |x, y\rangle & \end{array}$$

for $x, y \in \{0, 1\}$ and addition \oplus modulo 2.

Each qubit fragment is specified by a sub-signature of $\Sigma^{(2)}$ and equations on the free PROP it generates; the common unitary interpretation then restricts to this syntax. The index \bullet records the chosen fragment, while Σ_\bullet , \mathbf{P}_\bullet , \mathbf{C}_\bullet , \mathbf{QCirc}_\bullet , \mathbf{Qubit}_\bullet , and $\llbracket \cdot \rrbracket_\bullet$ denote its gates, free PROP, equations, presented PROP, generated semantic subPROP, and quotient interpretation, respectively.

► **Definition 15.** Let $\bullet \in \{\text{Cliff}, \text{RCliff}, \text{CliffT}, \text{CliffCS}, \text{CNOTdihe}\}$ range over the five qubit fragments considered in this paper. Given a sub-signature $\Sigma_\bullet \subseteq \Sigma^{(2)}$, let $\mathbf{P}_\bullet := \mathbf{P}_{\Sigma_\bullet}$ be the free PROP on Σ_\bullet . If \mathbf{C}_\bullet is a finite set of well-typed equations in \mathbf{P}_\bullet , define the presented PROP $\mathbf{QCirc}_\bullet := \mathbf{P}_\bullet / \mathbf{C}_\bullet$. The semantic target is the subPROP $\mathbf{Qubit}_\bullet := \langle \llbracket g \rrbracket^{(2)} \mid g \in \Sigma_\bullet \rangle \subseteq \mathbf{Qubit}$, generated by the interpretations of the gates in Σ_\bullet . When \mathbf{C}_\bullet is sound for the restricted common interpretation, the quotient carries the induced strict symmetric monoidal functor $\llbracket \cdot \rrbracket_\bullet : \mathbf{QCirc}_\bullet \rightarrow \mathbf{Qubit}_\bullet$. We call such a presentation complete when this functor is faithful.

► **Definition 16.** The five qubit fragments treated in this paper are specified by these sub-signatures and rule sets:

| Fragment | $\Sigma_\bullet \subseteq \Sigma^{(2)}$ | Rule set \mathbf{C}_\bullet |
|---------------|---|--|
| Clifford | $\Sigma_{\text{Cliff}} := \{\omega_8, \text{-H-}, \text{-S-}, \text{-CNOT-}\}$ | $\mathbf{C}_{\text{Cliff}} := \text{Cliff}_2$ (Figure 2) |
| Real Clifford | $\Sigma_{\text{RCliff}} := \{\ominus, \text{-H-}, \text{-Z-}, \text{-CNOT-}\}$ | $\mathbf{C}_{\text{RCliff}} := \text{RCliff}_2$ (Figure 3) |
| Clifford+T | $\Sigma_{\text{CliffT}} := \{\omega_8, \text{-H-}, \text{-T-}, \text{-CNOT-}\}$ | $\mathbf{C}_{\text{CliffT}} := \text{CliffT}_2$ (Figure 5) |
| Clifford+CS | $\Sigma_{\text{CliffCS}} := \{\omega_8, \text{-H-}, \text{-S-}, \text{-CNOT-}\}$ | $\mathbf{C}_{\text{CliffCS}} := \text{CliffCS}_2$ (Figure 6) |
| CNOT-dihedral | $\Sigma_{\text{CNOTdihe}} := \{\omega_8, \text{-X-}, \text{-T-}, \text{-CNOT-}\}$ | $\mathbf{C}_{\text{CNOTdihe}} := \text{CNOTdihe}_2$ (Figure 7) |

3.2 The qutrit Clifford fragment

For qutrits, the paper uses only the Clifford fragment, denoted Cliff_3 , whose signature is separate from the common qubit signature; we use the computational basis $\{|0\rangle, |1\rangle, |2\rangle\}$ of \mathbb{C}^3 and the induced tensor-product basis.

► **Definition 17.** The qutrit Clifford fragment is generated by the signature $\Sigma_{\text{Cliff}_3} := \{\omega_{12} : 0 \rightarrow 0, \text{-H-}, \text{-S-} : 1 \rightarrow 1, \text{-CNOT-} : 2 \rightarrow 2\}$.

Let $\mathbf{P}_{\text{Cliff}_3} := \mathbf{P}_{\Sigma_{\text{Cliff}_3}}$ be the free PROP on Σ_{Cliff_3} and let $C_{\text{Cliff}_3} := \text{Cliff}_3$ be the rule set given in Figure 4. The presented PROP is $\mathbf{QClirc}_{\text{Cliff}_3} := \mathbf{P}_{\text{Cliff}_3}/C_{\text{Cliff}_3}$.

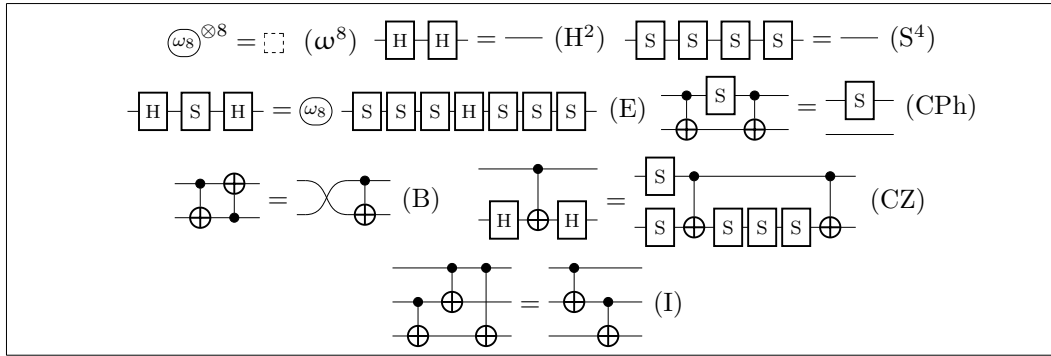
By the universal property of the free PROP, the assignments

$$\begin{aligned} \llbracket \omega_{12} \rrbracket_{\text{Cliff}_3} : 1 &\mapsto e^{i\pi/6} & \llbracket \text{H} \rrbracket_{\text{Cliff}_3} : |x\rangle &\mapsto \frac{1}{\sqrt{3}} \sum_{k=0}^2 e^{2\pi i x k/3} |k\rangle \\ \llbracket \text{CNOT} \rrbracket_{\text{Cliff}_3} : |x, y\rangle &\mapsto |x, (x+y) \bmod 3\rangle & \llbracket \text{S} \rrbracket_{\text{Cliff}_3} : |x\rangle &\mapsto e^{i\pi x(x-1)/3} |x\rangle \end{aligned}$$

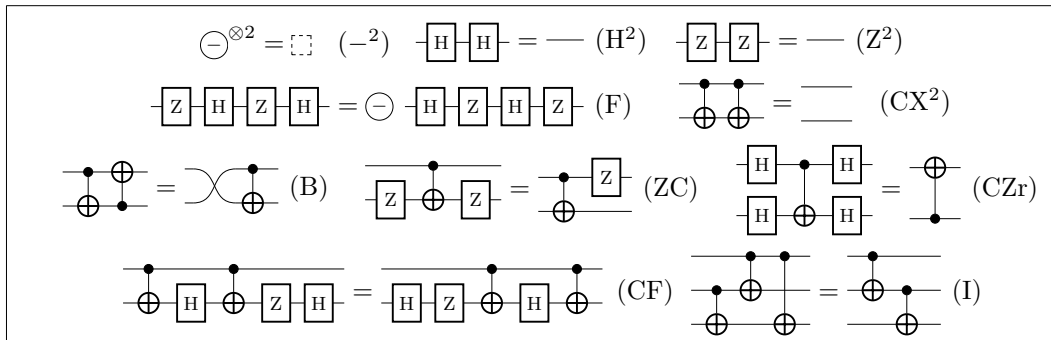
for $x, y \in \{0, 1, 2\}$ determine a strict symmetric monoidal functor $\mathbf{P}_{\text{Cliff}_3} \rightarrow \mathbf{Qutrit}$. Let $\mathbf{Qutrit}_{\text{Cliff}_3} := \langle \llbracket g \rrbracket_{\text{Cliff}_3} \mid g \in \Sigma_{\text{Cliff}_3} \rangle$ be the semantic subPROP generated by the qutrit Clifford gates. Since the equations of C_{Cliff_3} are sound for this interpretation, the free-PROP interpretation factors through the quotient and yields $\llbracket \cdot \rrbracket_{\text{Cliff}_3} : \mathbf{QClirc}_{\text{Cliff}_3} \rightarrow \mathbf{Qutrit}_{\text{Cliff}_3}$.

3.3 Rule-set presentations

Each displayed equation is an arity-preserving schema; gates outside the corresponding fragment signature are shortcuts expanded as in Figure 9.



■ **Figure 2** Qubit Clifford presentation Cliff_2 , with the scalar, one-qubit, controlled-phase, swap-decomposition, controlled-Z, and three-wire interaction laws used in the completeness and minimality arguments.



■ **Figure 3** Real Clifford presentation RCliff_2 , where the real phase, Hadamard-Z, controlled-X, controlled-Z, and three-wire interaction laws replace the corresponding Clifford rules in the real fragment.

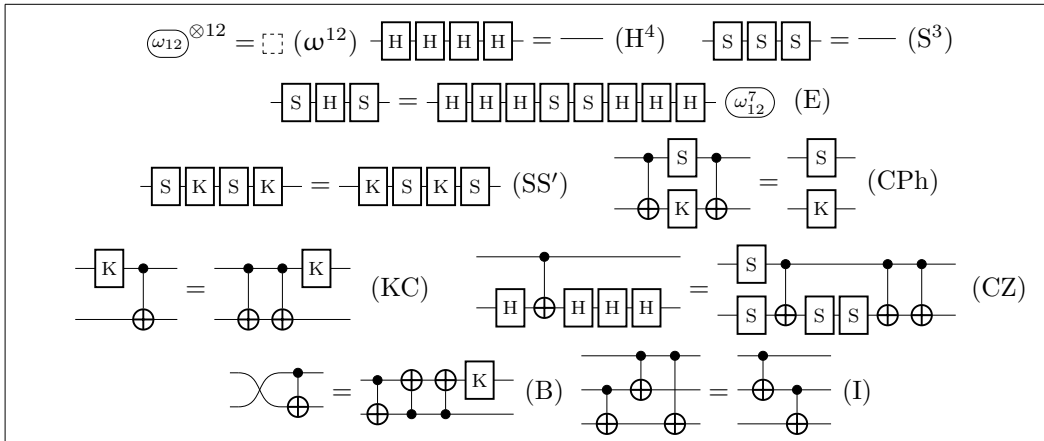


Figure 4 Qutrit Clifford presentation Cliff_3 , using the qutrit scalar convention and the qutrit versions of the phase, controlled-addition, controlled-Z, swap-decomposition, and three-wire laws.

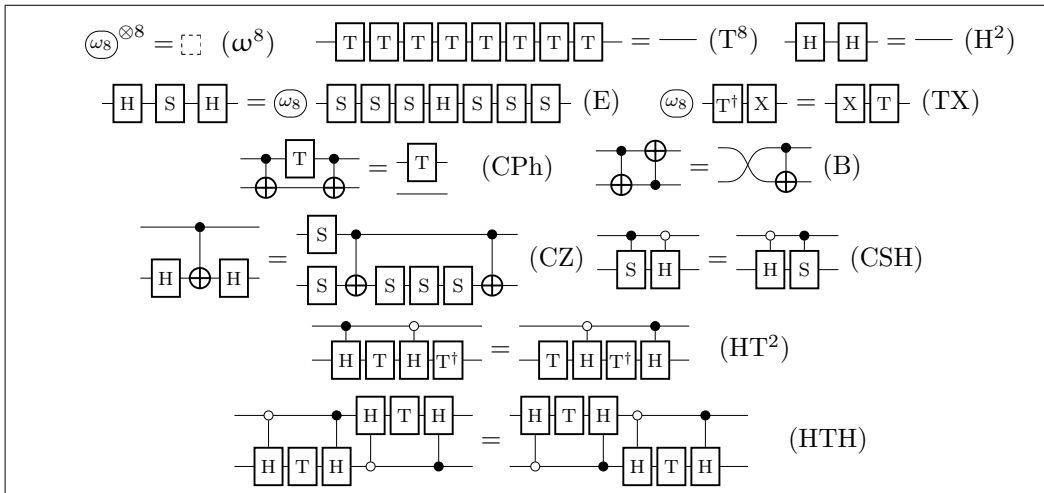
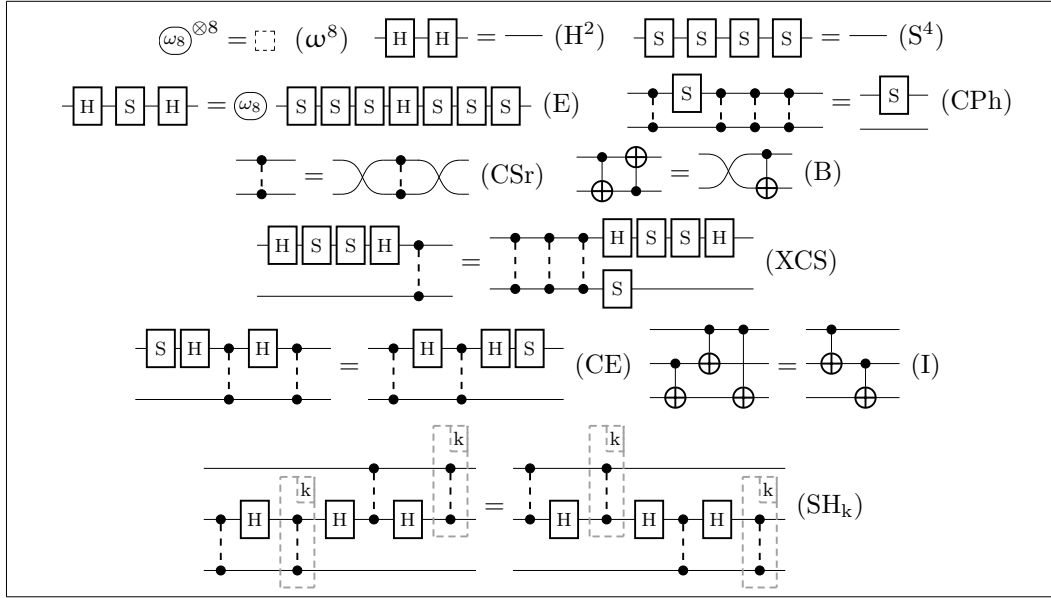


Figure 5 Qubit Clifford+ T presentation Cliff_{T_2} , combining the Clifford core with the T -phase, controlled-phase, controlled-Hadamard, and HTH -type relations needed for the two-qubit completeness transfer.

4 Completeness: from PROs to PROPs

For each fragment \bullet of Section 3 we have a presented PROP $\text{QCirc}_\bullet = \mathbf{P}_\bullet / \mathbf{C}_\bullet$ and a strict symmetric monoidal interpretation $\llbracket \cdot \rrbracket_\bullet : \text{QCirc}_\bullet \rightarrow \text{Qubit}_\bullet$ (or $\text{Qutrit}_{\text{Cliff}_3}$ in the qutrit case). Completeness means faithfulness of this interpretation. Complete equational theories for these fragments already exist in the literature [?, ?, ?, ?, ?, ?]. Those theorems supply the source completeness results: their normal forms prove faithfulness for older source presentations, and the transfer argument in this section checks that the source syntax and our smaller figure-defined target syntax present the same semantic subPROP.

The transfer has to align three conventions before the cited normal forms can be used for our quotient PROPs. If a source theorem is stated for a PRO, Lemma 10 adds the structural symmetries needed to read it as a PROP theorem. For qutrit Clifford and Clifford+ CS , the source and target scalar subgroups differ: the source presentations make fewer global phases explicit, which is harmless for the original proofs but does not match the standard strict matrices fixed in Section 3. The scalar-refinement step adjoins the missing roots of



■ **Figure 6** Qubit Clifford+ CS presentation CliffCS_2 , combining the Clifford phase laws with controlled- S commutation and the two three-wire relations used for the three-qubit transfer; in Equation (SH_k) , the dashed box repeats the enclosed subcircuit k times for $k \in \{0, 1, 2, 3\}$.

unity, so source and target have the same strict semantics. Encoding and decoding are then identity-on-objects PROP morphisms, in the sense of Definition 6, that compare the source generators and relations with ours; once these maps preserve semantics and decode source axioms to derivable target equations, Theorem 21 transports faithfulness.

4.1 Scalar refinement

Because our semantics uses strict unitary equality rather than projective equality, global phases are visible. Equivalently, the scalar subgroup $S(\mathbf{C}) = \mathbf{C}(0, 0)$ of the semantic PROP is part of the data that must match between source and target presentations. For our purposes, a scalar mismatch is therefore a mismatch between presentation conventions: the source convention makes fewer global phases explicit, while our convention keeps the standard gate matrices and records the extra phases as scalars.

Only two source completeness results require scalar adjustment. The qutrit Clifford presentation of [?] uses an order-6 scalar subgroup whereas our semantics uses order 12, and the Clifford+ CS presentation of [?] uses order 4 whereas our qubit semantics uses order 8. In both cases we appeal to the scalar-refinement construction of Appendix I, which adjoins the missing scalar generator and is conservative by Lemma 50. Source and target can then be compared inside the same semantic subPROP, while the refinement remains conservative on the old source signature.

4.2 Encoding, decoding, and transfer of completeness

The five fragments handled by encoding and decoding reduce to one generator-and-relation comparison.

Fix one encoding/decoding transfer instance. Here \mathbf{P} and \mathcal{R} denote the target free PROP and target relations, while \mathbf{P}^{src} and \mathcal{R}^{src} denote the corresponding source free PROP and source relations, after PRO-to-PROP lifting and scalar refinement where those steps apply.

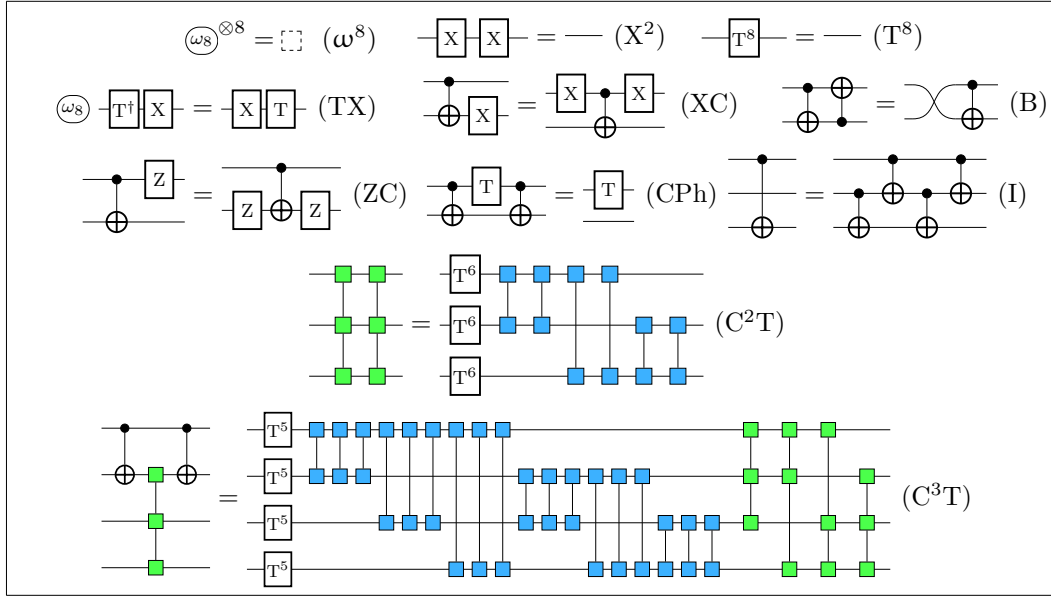


Figure 7 CNOT-dihedral presentation CNOTdihe_2 , with the X , T , CNOT, phase-commutation, C^2T , and C^3T relations used to recover the omitted source axioms.

► **Definition 18.** An encoding/decoding pair consists of a target-to-source PROP morphism $E : \mathbf{P} \rightarrow \mathbf{P}^{\text{src}}$ and a source-to-target PROP morphism $D : \mathbf{P}^{\text{src}} \rightarrow \mathbf{P}$ such that $\llbracket C \rrbracket = \llbracket E(C) \rrbracket_{\text{src}}$ for every target circuit $C \in \mathbf{P}$ and $\llbracket C' \rrbracket_{\text{src}} = \llbracket D(C') \rrbracket$ for every source circuit $C' \in \mathbf{P}^{\text{src}}$.

After importing a source completeness theorem, the comparison is local: it involves only the chosen generators and source relations. A typical mismatch is that a source presentation takes controlled- Z as primitive, whereas our target signature takes CNOT. The decoder replaces source controlled- Z by its fixed expansion through CNOT and Hadamards; the target-generator check then proves, for example, that decoding the encoded CNOT derives CNOT again, while the source-relation check verifies each source equation after the same replacement. The non-identity parts of the maps used here are recorded in Table 3.

The transfer therefore has a finite checklist: the round-trip equality $D(E(g)) = g$ for each target generator g , and the decoded equality $D(L) = D(R)$ for each source axiom $L = R$. This is the presentation-level analogue of Reidemeister–Schreier rewriting [?, ?]: instead of computing a subgroup presentation, we compare two chosen presentations by following the images of their generators and defining relations.

► **Lemma 19.** Assume \mathbf{P} is presented by generators Σ and relations \mathcal{R} . If for every non-structural generator $g \in \Sigma$, $\mathbf{P}/\mathcal{R} \vdash D(E(g)) = g$, then for every circuit C built from Σ , $\mathbf{P}/\mathcal{R} \vdash D(E(C)) = C$.

Proof. By structural induction on C , using that E and D preserve \circ , \otimes , and the structural morphisms. ◀

► **Lemma 20.** Assume that for every axiom $L = R$ in \mathcal{R}^{src} , $\mathbf{P}/\mathcal{R} \vdash D(L) = D(R)$. Then D induces a PROP morphism $\bar{D} : \mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}} \rightarrow \mathbf{P}/\mathcal{R}$, and any derivable equality in the source quotient transports to a derivable equality between its decodings in the target theory.

Proof. The hypothesis says exactly that D equalises all source relations, hence factors through the quotient. ◀

► **Theorem 21.** *Assume $\mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}}$ is complete for $\llbracket \cdot \rrbracket_{\text{src}}$, and let (E, D) be an encoding/decoding pair satisfying the hypotheses of Lemmas 19 and 20. Then \mathbf{P}/\mathcal{R} is complete for $\llbracket \cdot \rrbracket$.*

Proof. If target circuits C_1, C_2 have the same semantics, then semantic preservation of E gives $\llbracket E(C_1) \rrbracket_{\text{src}} = \llbracket E(C_2) \rrbracket_{\text{src}}$. Source completeness yields $E(C_1) = E(C_2)$ in $\mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}}$; applying the quotient morphism \overline{D} from Lemma 20 gives $D(E(C_1)) = D(E(C_2))$ in \mathbf{P}/\mathcal{R} , and Lemma 19 rewrites both sides to $C_1 = C_2$. ◀

The same proof gives the bounded form used in Theorem 22: if the source interpretation is faithful on all hom-sets $n \rightarrow m$ with $n, m \leq k$, then the target interpretation is faithful on the same range, because E and D are identity-on-objects PROP morphisms.

4.3 Fragment-wise completeness

Applying the transfer to the five encoding/decoding fragments and the direct CNOT-dihedral argument gives the completeness bounds used by the minimality theorem.

- **Theorem 22.** 1. $\mathbf{QCirc}_{\text{Cliff}}$ is complete for $\mathbf{Qubit}_{\text{Cliff}}$.
- 2. $\mathbf{QCirc}_{\text{RCliff}}$ is complete for $\mathbf{Qubit}_{\text{RCliff}}$.
- 3. $\mathbf{QCirc}_{\text{Cliff3}}$ is complete for $\mathbf{Qutrit}_{\text{Cliff3}}$.
- 4. $\mathbf{QCirc}_{\text{CliffT}}$ is complete up to 2 qubits for $\mathbf{Qubit}_{\text{CliffT}}$.
- 5. $\mathbf{QCirc}_{\text{CliffCS}}$ is complete up to 3 qubits for $\mathbf{Qubit}_{\text{CliffCS}}$.
- 6. $\mathbf{QCirc}_{\text{CNOTdih}}$ is complete for $\mathbf{Qubit}_{\text{CNOTdih}}$.

Proof. For the five fragments using encoding/decoding, start from the corresponding cited source completeness theorem; in the qutrit Clifford and Clifford+CS cases, first pass to the scalar-refined source presentation supplied by Lemma 50. The arity bounds in the Clifford+T and Clifford+CS items are the bounds inherited from those source results.

The appendices record the data needed to invoke Theorem 21: the earlier rule sets in Figures 12–17, the encoding/decoding maps in Table 3, the generator checks for Lemma 19 in Appendix F, and the source-axiom checks for Lemma 20 in Appendix G. With these ingredients fixed, the remaining derivability obligations are exactly the finite checks of the transfer lemmas; no further normal-form argument is used here.

For CNOT-dihedral, use the completeness theorem of [?] directly: no encoding/decoding is needed, and Appendix H derives, in CNOTdih₂, each source axiom removed from our simplified presentation. ◀

5 Minimality

Minimality asks which axioms of the six figure-defined presentations are genuinely necessary for completeness, and the full and bounded statements in Theorem 37 are certified by separating interpretations. Throughout, fix a fragment \bullet with free PROP \mathbf{P}_\bullet and finite rule set \mathcal{C}_\bullet . The arity bounds in the theorem are dictated by separator coverage: a bounded minimality claim is made only when the table contains a non-none separator for every axiom in that truncation. When this independence is read as necessity for completeness, the Clifford+T and Clifford+CS cases are understood relative to the bounded completeness results imported in Section 4.

5.1 Independence and minimality

To test whether an axiom ρ is needed, remove it from the rule set and ask whether its two sides are still equal in the quotient by the remaining rules. We write $C_\bullet \setminus \{\rho\}$ for those remaining axioms, and $\mathbf{QCirc}_\bullet^{-\rho} := \mathbf{P}_\bullet / (C_\bullet \setminus \{\rho\})$ for the reduced presentation.

► **Definition 23.** *Let ρ be an axiom $C_1 = C_2$ in C_\bullet (with $C_1, C_2 : n \rightarrow n$). We say that ρ is independent of the other axioms of the same fragment if $\mathbf{QCirc}_\bullet^{-\rho} \not\vdash C_1 = C_2$. We say that C_\bullet is minimal if every $\rho \in C_\bullet$ is independent.*

Because the presentations are sound for their intended semantics, an axiom derivable from the others cannot be necessary for completeness: removing it leaves the induced congruence unchanged. Conversely, in any arity range where the reduced presentation were still complete, soundness would force the removed equation to remain derivable. Thus independence is the syntactic test used here, and the separators in Figure 8 instantiate it by occurrence detectors, counting models, projective substitutions, and determinant phases.

Since derivations preserve arity by Remark 14, an n -wire axiom can be tested inside the subsystem generated by axioms of arity at most n . This gives the bounded notion used when the separator table contains separators only in a finite arity range.

► **Definition 24.** *Let $k \in \mathbb{N}$. Define $C_{\bullet, \leq k} := \{\rho \in C_\bullet \mid \rho \text{ has type } n \rightarrow n \text{ with } n \leq k\}$. We say that C_\bullet is minimal up to k wires if every $\rho \in C_{\bullet, \leq k}$ is independent relative to $C_{\bullet, \leq k}$ (i.e. independent in the sense of Definition 23 after replacing C_\bullet by $C_{\bullet, \leq k}$).*

5.2 Separation by alternative interpretations

Our independence proofs are semantic: to show that an axiom ρ is not derivable from the others, we construct a PROP-valued model that satisfies all remaining axioms but not ρ . The underlying countermodel principle for equational logic is that derivable equations hold in every model of the theory, so one model of the reduced theory that violates ρ rules out a derivation. This principle goes back at least to Birkhoff's correspondence between equational theories and varieties of algebras [?] and is explicit in modern accounts of equational completeness [?, Thm. 14.19]. In the quantum-circuit literature, the same proof pattern is used in [?], which proves a minimal and complete equational theory for general quantum circuits.

► **Lemma 25.** *Let $\rho : C_1 = C_2$ be an axiom in C_\bullet . Assume there exist a PROP \mathbf{P}_{alt} and a PROP morphism $F : \mathbf{P}_\bullet \rightarrow \mathbf{P}_{\text{alt}}$ such that $F(L) = F(R)$ for every $(L = R) \in C_\bullet \setminus \{\rho\}$, but $F(C_1) \neq F(C_2)$. Then ρ is independent.*

Proof. Since F equalises every axiom in $C_\bullet \setminus \{\rho\}$, it factors through the quotient $\mathbf{QCirc}_\bullet^{-\rho}$, yielding a PROP morphism $\bar{F} : \mathbf{QCirc}_\bullet^{-\rho} \rightarrow \mathbf{P}_{\text{alt}}$. If $\mathbf{QCirc}_\bullet^{-\rho} \vdash C_1 = C_2$ then applying \bar{F} gives $F(C_1) = F(C_2)$, a contradiction. ◀

Because morphisms of the free PROP are already quotiented by coherence, once F is a PROP morphism it automatically respects all structural equalities, so the separator check is purely fragment-specific.

5.3 Families of separating interpretations

Figure 8 is the finite certificate for the independence proofs: each non-none row supplies one separator to be checked against the other axioms in the relevant arity range, while the none rows mark exactly the gaps that prevent full minimality claims for qutrit Clifford, Clifford+T,

and Clifford+CS. The separators fall into four families—counting models, occurrence detectors, projective substitutions, and determinant-phase interpretations—so the proof remains uniform even though the fragments differ. In displayed rule sets and tables, abbreviated gates stand for fixed morphisms of the free PROP, and every separating interpretation is applied to the expanded morphisms.

Counting and occurrence detectors. Counting models record selected generator multiplicities or swap parity, while occurrence detectors record whether a chosen generator appears; both kinds of separator land in endomorphism-only PROPs, so there are no morphisms between different arities.

► **Definition 26.** Let $(M, \oplus, 0)$ be a commutative monoid and let $\epsilon \in M$ satisfy $\epsilon \oplus \epsilon = 0$. Define $\mathbf{End}_\epsilon(M)$ to be the PROP with objects \mathbb{N} and hom-sets $\mathbf{End}_\epsilon(M)(n, n) := M$ ($n \in \mathbb{N}$), $\mathbf{End}_\epsilon(M)(n, m) := \emptyset$ ($n \neq m$). Composition and tensor are given by \oplus , identities by 0, and the basic symmetry $\sigma_{1,1}$ is interpreted as ϵ .

► **Remark 27.** The condition $\epsilon \oplus \epsilon = 0$ implies $\sigma_{m,n} \circ \sigma_{n,m} = \text{id}_{n+m}$. All other PROP coherence axioms hold because every structural map is interpreted as an element of the commutative monoid M , and both \circ and \otimes are interpreted as \oplus .

► **Definition 28.** For $m \geq 2$ define $\mathbf{Count}_m := \mathbf{End}_0(\mathbb{Z}_m, +, 0)$ and $\mathbf{Has} := \mathbf{End}_0(\mathbb{B}, \vee, 0)$, with $\mathbb{B} = \{0, 1\}$. When we want to record the parity of the structural swap, we also use $\mathbf{Count}_2^{\text{swap}} := \mathbf{End}_1(\mathbb{Z}_2, +, 0)$, so that the basic symmetry $\sigma_{1,1}$ is interpreted as $1 \in \mathbb{Z}_2$.

► **Definition 29.** Let $m \geq 2$. Given a finite list (g_1, \dots, g_r) of generators in Σ_\bullet , with repetitions counted with multiplicity, write $\#\{g_1, \dots, g_r\}_{[m]} : \mathbf{P}_\bullet \rightarrow \mathbf{Count}_m$ for the unique PROP morphism sending each generator g to the number of its occurrences in the list modulo m .

When $m = 2$, we use the symbol \bowtie inside the braces only as a mnemonic that swaps are counted: $\#\{g_1, \dots, g_r, \bowtie\}_{[2]} : \mathbf{P}_\bullet \rightarrow \mathbf{Count}_2^{\text{swap}}$. In particular, $\#\{\bowtie\}_{[2]}$ denotes the morphism that sends every non-structural generator to 0 and counts each structural swap as 1.

► **Definition 30.** For a generator $g \in \Sigma_\bullet$, write $?g : \mathbf{P}_\bullet \rightarrow \mathbf{Has}$ for the unique PROP morphism with $?g(g) = 1$ and $?g(h) = 0$ for every generator $h \neq g$. Equivalently, for any circuit C , one has $?g(C) = 1$ if and only if g occurs at least once in C .

► **Example 31.** In the one-wire Clifford+T truncation, the detector $?-\boxed{\text{H}}-$ separates the axiom (H^2) from the remaining rules. The equations (ω^8) and (T^8) contain no $-\boxed{\text{H}}-$ on either side, while (E) contains an occurrence of $-\boxed{\text{H}}-$ on both sides. The same is true of (TX) after expanding the displayed shortcut $-\boxed{\text{X}}-$. Thus $?-\boxed{\text{H}}-$ sends each remaining equation in $\mathbf{C}_{\text{CliffT}, \leq 1}$ to either $0 = 0$ or $1 = 1$. On the removed axiom, however, $?-\boxed{\text{H}}-(\boxed{\text{H}}-\boxed{\text{H}}-) = 1$ and $?-\boxed{\text{H}}-(\text{id}_1) = 0$, so (H^2) is independent by Lemma 25.

Projective substitutions. These separators alter one generator while quotienting only by global phase, which separates rows whose two sides have the same generator multiplicities but differ after a projective change of one generator.

► **Definition 32.** For $d \in \{2, 3\}$, let \mathbf{Qudit}_d^\sim be the projective quotient of \mathbf{Qudit}_d from Definition 12: its endomorphism hom-set is $\mathbf{Qudit}_d(n, n)/\text{U}(1)$, where $\text{U}(1)$ acts by global phase, and its non-endomorphism hom-sets remain empty. Write $\Pi_d : \mathbf{Qudit}_d \rightarrow \mathbf{Qudit}_d^\sim$ for the quotient PROP morphism.

► **Definition 33.** For a fragment \bullet , set $d(\bullet) = 3$ for the qutrit Clifford fragment and $d(\bullet) = 2$ for the qubit fragments. Its usual projective interpretation on the free presentation is $\llbracket \cdot \rrbracket_{\bullet, \sim} := \Pi_{d(\bullet)} \circ \llbracket \cdot \rrbracket_{\bullet}^{\text{free}}$, regarded as a PROP morphism $\mathbf{P}_{\bullet} \rightarrow \mathbf{Qudit}_{d(\bullet)}^{\sim}$, where $\llbracket \cdot \rrbracket_{\bullet}^{\text{free}} : \mathbf{P}_{\bullet} \rightarrow \mathbf{Qudit}_{d(\bullet)}$ is the standard unitary interpretation before quotienting by the fragment equations.

► **Definition 34.** Given a fragment \bullet , a generator $g : n \rightarrow n$, and a circuit $C : n \rightarrow n$ of that fragment, write $\llbracket \cdot \rrbracket_{\bullet, \sim}^{g:=C}$ for the unique PROP morphism $\mathbf{P}_{\bullet} \rightarrow \mathbf{Qudit}_{d(\bullet)}^{\sim}$ sending g to the already-evaluated class $\llbracket C \rrbracket_{\bullet, \sim}$ and every generator $h \neq g$ to $\llbracket h \rrbracket_{\bullet, \sim}$. Displayed abbreviations in C are expanded before this interpretation.

Thus a table entry $[g := C]_{\sim}$ means that every generator is interpreted projectively as usual except g , whose value is replaced by the projective value of C .

Determinant phases. These separators record a scaled determinant phase for the qubit rows, chosen so as to be strictly monoidal on a bounded number of wires.

► **Definition 35.** Fix a qubit fragment, let $k \geq 2$, and write $\arg \det(U) \in \mathbb{R}/2\pi\mathbb{Z}$ for the phase of $\det(U)$. Set $\epsilon_k := 2^{k-2}\pi \bmod 2\pi$, so $\epsilon_2 = \pi$ and $\epsilon_k = 0$ for $k > 2$.

Define a PROP morphism $\arg \det_k : \mathbf{P}_{\bullet} \rightarrow \mathbf{End}_{\epsilon_k}(\mathbb{R}/2\pi\mathbb{Z}, +, 0)$ on generators $g : n \rightarrow n$ by $(\arg \det_k)(g) := 2^{k-n} \arg \det(\llbracket g \rrbracket) \bmod 2\pi$, and extend it to all circuits using preservation of \circ , \otimes , and symmetries.

► **Remark 36.** The scaling factor 2^{k-n} is chosen so that $\arg \det_k$ is compatible with tensoring on arities $\leq k$, using $\det(U \otimes V) = \det(U)^{2^m} \det(V)^{2^n}$ for U on n qubits and V on m qubits. The choice of ϵ_k makes the image of the structural swap $\sigma_{1,1}$ coincide with its determinant phase after scaling: it contributes π when $k = 2$ and becomes 0 for $k > 2$. For equations on more than k wires, $\arg \det_k$ is still used only as the generator-defined PROP morphism above; the row-wise separator check is a check in $\mathbf{End}_{\epsilon_k}(\mathbb{R}/2\pi\mathbb{Z}, +, 0)$, not an assertion that the displayed value is the scaled determinant of the whole higher-arity unitary.

For each non-none row of Figure 8, the relation named on the left is removed and the separator on the right equalises the other axioms in the relevant arity range while distinguishing the removed relation. The table records the separator rather than every value in the finite check. Detector and counting rows use the relevant commutative monoid, projective substitution rows use the projective quotient from Definition 33, and determinant rows use the scaled determinant phase from Definition 35. When a higher-arity separator is used in Theorem 37, its check is against all axioms in the relevant arity truncation, including same-arity rows marked none in Figure 8.

5.4 Minimality statements

The none rows lie outside the bounded ranges claimed in Theorem 37. When the table contains a separator beyond such a range, the corresponding higher-arity axiom is stated separately.

► **Theorem 37.** The indicated presented circuit fragments have minimal rule sets as listed:

1. Cliff_2 is minimal for $\mathbf{P}_{\text{Cliff}}$ -circuits (all arities).
2. RCliff_2 is minimal for $\mathbf{P}_{\text{RCliff}}$ -circuits (all arities).
3. Cliff_3 is minimal for $\mathbf{P}_{\text{Cliff}_3}$ -circuits up to 2 qutrit wires.¹

¹ Full minimality would follow from a separator for the 3-qutrit axiom Equation (I).

| (a) Clifford | (b) Real Clifford | (c) Qutrit Clifford | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|--|---|----------------------------------|--------------|----------------|---------|----------------|---------|----------------|-----|------------------------|-------|--|-------|------------------------|------|----------------------------------|------|----------------------------------|--|------|-----------|---------|---------------|---------|--|---------|----------------|--------------|-----------------------|----------|----------------|---------|------------------------|------|------------------------|-------|---------------------------------|-------|---------------------------------|-------|----------------------------------|--|---|-----------|---|-------------------|--------------------|--------------------|---------|--|------|------------------------|--------------|---|---------|----------------|---------|------------------------|------|------------------------|------|------------------------|-------|--------------|-----|------------------------|------|--|-----|--------------------------------|----------|--|----------|--------------------|
| <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Rel.</th> <th>Separator</th> </tr> </thead> <tbody> <tr><td>(ω^8)</td><td>$?_{\omega^8}$</td></tr> <tr><td>(H^2)</td><td>$?_{\text{H}}$</td></tr> <tr><td>(S^4)</td><td>$?_{\text{S}}$</td></tr> <tr><td>(E)</td><td>$\#\{\text{H}\}_{[2]}$</td></tr> <tr><td>(CPh)</td><td>$?_{\oplus}$</td></tr> <tr><td>(B)</td><td>$\#\{\text{C}\}_{[2]}$</td></tr> <tr><td>(CZ)</td><td>$\#\{\text{C}, \text{X}\}_{[2]}$</td></tr> <tr><td>(I)</td><td>arg det_2</td></tr> </tbody> </table> | Rel. | Separator | (ω^8) | $?_{\omega^8}$ | (H^2) | $?_{\text{H}}$ | (S^4) | $?_{\text{S}}$ | (E) | $\#\{\text{H}\}_{[2]}$ | (CPh) | $?_{\oplus}$ | (B) | $\#\{\text{C}\}_{[2]}$ | (CZ) | $\#\{\text{C}, \text{X}\}_{[2]}$ | (I) | arg det_2 | <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Rel.</th> <th>Separator</th> </tr> </thead> <tbody> <tr><td>$(-^2)$</td><td>$?_{\ominus}$</td></tr> <tr><td>(H^2)</td><td>$?_{\text{H}}$</td></tr> <tr><td>(Z^2)</td><td>$?_{\text{Z}}$</td></tr> <tr><td>(F)</td><td>$\#\{\ominus\}_{[2]}$</td></tr> <tr><td>(CX^2)</td><td>$?_{\oplus}$</td></tr> <tr><td>(B)</td><td>$\#\{\text{C}\}_{[2]}$</td></tr> <tr><td>(ZC)</td><td>$\#\{\text{Z}\}_{[2]}$</td></tr> <tr><td>(CF)</td><td>$[\text{Z}] := \text{---} \sim$</td></tr> <tr><td>(CZr)</td><td>$[\text{H}] := \text{---} \sim$</td></tr> <tr><td>(I)</td><td>arg det_2</td></tr> </tbody> </table> | Rel. | Separator | $(-^2)$ | $?_{\ominus}$ | (H^2) | $?_{\text{H}}$ | (Z^2) | $?_{\text{Z}}$ | (F) | $\#\{\ominus\}_{[2]}$ | (CX^2) | $?_{\oplus}$ | (B) | $\#\{\text{C}\}_{[2]}$ | (ZC) | $\#\{\text{Z}\}_{[2]}$ | (CF) | $[\text{Z}] := \text{---} \sim$ | (CZr) | $[\text{H}] := \text{---} \sim$ | (I) | arg det_2 | <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Rel.</th> <th>Separator</th> </tr> </thead> <tbody> <tr><td>(ω^{12})</td><td>$?_{\omega^{12}}$</td></tr> <tr><td>(H^4)</td><td>$?_{\text{H}}$</td></tr> <tr><td>(S^3)</td><td>$?_{\text{S}}$</td></tr> <tr><td>(E)</td><td>$\#\{\text{H}\}_{[2]}$</td></tr> <tr><td>(SS')</td><td>$[\text{S}] := [\text{S}][\text{X}] \sim^a$</td></tr> <tr><td>(CPh)</td><td>$?_{\oplus}$</td></tr> <tr><td>(B)</td><td>$\#\{\text{C}\}_{[2]}$</td></tr> <tr><td>(CZ)</td><td>$\#\{\text{S}\}_{[3]}$</td></tr> <tr><td>(KC)</td><td>$\#\{\oplus\}_{[2]}$</td></tr> <tr><td>(I)</td><td>none</td></tr> </tbody> </table> | Rel. | Separator | (ω^{12}) | $?_{\omega^{12}}$ | (H^4) | $?_{\text{H}}$ | (S^3) | $?_{\text{S}}$ | (E) | $\#\{\text{H}\}_{[2]}$ | (SS') | $[\text{S}] := [\text{S}][\text{X}] \sim^a$ | (CPh) | $?_{\oplus}$ | (B) | $\#\{\text{C}\}_{[2]}$ | (CZ) | $\#\{\text{S}\}_{[3]}$ | (KC) | $\#\{\oplus\}_{[2]}$ | (I) | none | | | | | | | | | | |
| Rel. | Separator | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ω^8) | $?_{\omega^8}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (H^2) | $?_{\text{H}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (S^4) | $?_{\text{S}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (E) | $\#\{\text{H}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CPh) | $?_{\oplus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (B) | $\#\{\text{C}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CZ) | $\#\{\text{C}, \text{X}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (I) | arg det_2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Rel. | Separator | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| $(-^2)$ | $?_{\ominus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (H^2) | $?_{\text{H}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (Z^2) | $?_{\text{Z}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (F) | $\#\{\ominus\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CX^2) | $?_{\oplus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (B) | $\#\{\text{C}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ZC) | $\#\{\text{Z}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CF) | $[\text{Z}] := \text{---} \sim$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CZr) | $[\text{H}] := \text{---} \sim$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (I) | arg det_2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Rel. | Separator | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ω^{12}) | $?_{\omega^{12}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (H^4) | $?_{\text{H}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (S^3) | $?_{\text{S}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (E) | $\#\{\text{H}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (SS') | $[\text{S}] := [\text{S}][\text{X}] \sim^a$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CPh) | $?_{\oplus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (B) | $\#\{\text{C}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CZ) | $\#\{\text{S}\}_{[3]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (KC) | $\#\{\oplus\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (I) | none | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | ^a See Proposition 38. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Rel.</th> <th>Separator</th> </tr> </thead> <tbody> <tr><td>(ω^8)</td><td>$?_{\omega^8}$</td></tr> <tr><td>(H^2)</td><td>$?_{\text{H}}$</td></tr> <tr><td>(T^8)</td><td>$?_{\text{T}}$</td></tr> <tr><td>(E)</td><td>$\#\{\text{H}\}_{[2]}$</td></tr> <tr><td>(TX)</td><td>$\#\{\text{H}, \text{T}, \omega^8\}_{[2]}$</td></tr> <tr><td>(CPh)</td><td>$?_{\oplus}$</td></tr> <tr><td>(B)</td><td>$\#\{\text{C}\}_{[2]}$</td></tr> <tr><td>(CZ)</td><td>$\#\{\text{C}, \text{X}\}_{[2]}$</td></tr> <tr><td>(CSH)</td><td>none</td></tr> <tr><td>(HT^2)</td><td>none</td></tr> <tr><td>(HTH)</td><td>none</td></tr> </tbody> </table> | Rel. | Separator | (ω^8) | $?_{\omega^8}$ | (H^2) | $?_{\text{H}}$ | (T^8) | $?_{\text{T}}$ | (E) | $\#\{\text{H}\}_{[2]}$ | (TX) | $\#\{\text{H}, \text{T}, \omega^8\}_{[2]}$ | (CPh) | $?_{\oplus}$ | (B) | $\#\{\text{C}\}_{[2]}$ | (CZ) | $\#\{\text{C}, \text{X}\}_{[2]}$ | (CSH) | none | (HT^2) | none | (HTH) | none | <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Rel.</th> <th>Separator</th> </tr> </thead> <tbody> <tr><td>(ω^8)</td><td>$?_{\omega^8}$</td></tr> <tr><td>(H^2)</td><td>$?_{\text{H}}$</td></tr> <tr><td>(S^4)</td><td>$?_{\text{S}}$</td></tr> <tr><td>(E)</td><td>$\#\{\text{H}\}_{[2]}$</td></tr> <tr><td>(CPh)</td><td>$?_{\oplus}$</td></tr> <tr><td>(B)</td><td>$\#\{\text{C}\}_{[2]}$</td></tr> <tr><td>(XCS)</td><td>$\#\{\text{S}, \text{H}\}_{[2]}$</td></tr> <tr><td>(CSr)</td><td>$[\text{S}] := [\text{S}][\text{Z}] \sim$</td></tr> <tr><td>(CE)</td><td>$[\text{H}] := [\text{H}][\text{Z}] \sim$</td></tr> <tr><td>(I)</td><td>arg det_2</td></tr> <tr><td>(SH_k)</td><td>none</td></tr> </tbody> </table> | Rel. | Separator | (ω^8) | $?_{\omega^8}$ | (H^2) | $?_{\text{H}}$ | (S^4) | $?_{\text{S}}$ | (E) | $\#\{\text{H}\}_{[2]}$ | (CPh) | $?_{\oplus}$ | (B) | $\#\{\text{C}\}_{[2]}$ | (XCS) | $\#\{\text{S}, \text{H}\}_{[2]}$ | (CSr) | $[\text{S}] := [\text{S}][\text{Z}] \sim$ | (CE) | $[\text{H}] := [\text{H}][\text{Z}] \sim$ | (I) | arg det_2 | (SH _k) | none | <table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th>Rel.</th> <th>Separator</th> </tr> </thead> <tbody> <tr><td>(ω^8)</td><td>$?_{\omega^8}$</td></tr> <tr><td>(T^8)</td><td>$?_{\text{T}}$</td></tr> <tr><td>(X^2)</td><td>$?_{\text{X}}$</td></tr> <tr><td>(TX)</td><td>$\#\{\omega^8\}_{[2]}$</td></tr> <tr><td>(XC)</td><td>$\#\{\text{X}\}_{[2]}$</td></tr> <tr><td>(CPh)</td><td>$?_{\oplus}$</td></tr> <tr><td>(B)</td><td>$\#\{\text{C}\}_{[2]}$</td></tr> <tr><td>(ZC)</td><td>$\#\{\text{T}, \omega^8, \omega^8\}_{[8]}$</td></tr> <tr><td>(I)</td><td>$\#\{\oplus, \text{X}\}_{[2]}$</td></tr> <tr><td>$(C^2T)$</td><td>$\#\{\text{T}, \omega^8, \omega^8\}_{[4]}$</td></tr> <tr><td>$(C^3T)$</td><td>$\text{arg det}_3$</td></tr> </tbody> </table> | Rel. | Separator | (ω^8) | $?_{\omega^8}$ | (T^8) | $?_{\text{T}}$ | (X^2) | $?_{\text{X}}$ | (TX) | $\#\{\omega^8\}_{[2]}$ | (XC) | $\#\{\text{X}\}_{[2]}$ | (CPh) | $?_{\oplus}$ | (B) | $\#\{\text{C}\}_{[2]}$ | (ZC) | $\#\{\text{T}, \omega^8, \omega^8\}_{[8]}$ | (I) | $\#\{\oplus, \text{X}\}_{[2]}$ | (C^2T) | $\#\{\text{T}, \omega^8, \omega^8\}_{[4]}$ | (C^3T) | arg det_3 |
| Rel. | Separator | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ω^8) | $?_{\omega^8}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (H^2) | $?_{\text{H}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (T^8) | $?_{\text{T}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (E) | $\#\{\text{H}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (TX) | $\#\{\text{H}, \text{T}, \omega^8\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CPh) | $?_{\oplus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (B) | $\#\{\text{C}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CZ) | $\#\{\text{C}, \text{X}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CSH) | none | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (HT^2) | none | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (HTH) | none | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Rel. | Separator | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ω^8) | $?_{\omega^8}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (H^2) | $?_{\text{H}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (S^4) | $?_{\text{S}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (E) | $\#\{\text{H}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CPh) | $?_{\oplus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (B) | $\#\{\text{C}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (XCS) | $\#\{\text{S}, \text{H}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CSr) | $[\text{S}] := [\text{S}][\text{Z}] \sim$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CE) | $[\text{H}] := [\text{H}][\text{Z}] \sim$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (I) | arg det_2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (SH _k) | none | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| Rel. | Separator | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ω^8) | $?_{\omega^8}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (T^8) | $?_{\text{T}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (X^2) | $?_{\text{X}}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (TX) | $\#\{\omega^8\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (XC) | $\#\{\text{X}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (CPh) | $?_{\oplus}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (B) | $\#\{\text{C}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (ZC) | $\#\{\text{T}, \omega^8, \omega^8\}_{[8]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (I) | $\#\{\oplus, \text{X}\}_{[2]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (C^2T) | $\#\{\text{T}, \omega^8, \omega^8\}_{[4]}$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| (C^3T) | arg det_3 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

■ **Figure 8** Separators witnessing independence, indexed by the equation labels from Section 3; entries marked none are the cases for which no separator is currently provided.

4. CliffT_2 is minimal for $\mathbf{P}_{\text{CliffT}}$ -circuits up to 1 qubit wire.
5. CliffCS_2 is minimal for $\mathbf{P}_{\text{CliffCS}}$ -circuits up to 2 qubit wires.
6. CNOTdih_2 is minimal for $\mathbf{P}_{\text{CNOTdih}}$ -circuits (all arities).

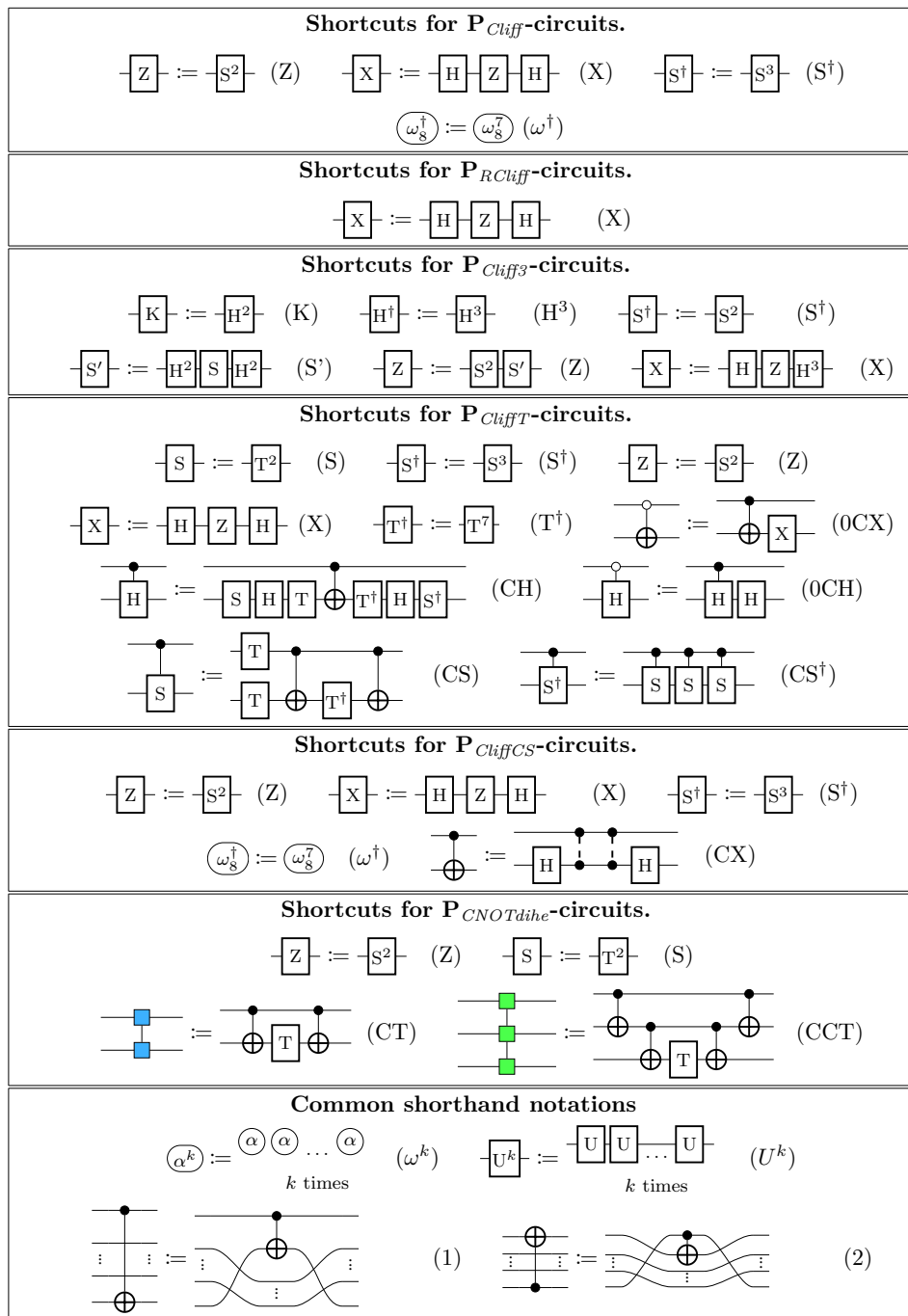
For the bounded fragments, the constructed higher-arity separators give:

- In CliffT_2 , the 2-qubit axioms (CPh), (B) and (CZ) are independent.
- In CliffCS_2 , the 3-qubit axiom (I) is independent.

Proof. For each axiom ρ covered by the statement, the corresponding non-none row of Figure 8 gives a separator F_ρ that equalises the remaining axioms of the relevant truncated system while distinguishing the two sides of ρ . For the additional higher-arity statements, the relevant truncated system is the one at the arity of the displayed axiom. Apply Lemma 25 and Definition 24. ◀

6 Conclusion

By placing the six fragments in a common PROP syntax with structural symmetries, the paper isolates fewer non-structural axioms while preserving the strict unitary semantics inherited from the source presentations. The completeness transfer and separator table show that this simplification often preserves irredundancy: qubit Clifford, real Clifford, and CNOT-dihedral are minimal in all arities, while the other fragments are minimal in the bounded ranges certified in Section 5. The none rows of Figure 8 identify the remaining separator-construction problem, and the reduced presentations give mechanised rewriting a smaller algebraic core: related work on 2-qubit Clifford+ T and 3-qubit Clifford+ CS already gives Agda checks for this style of argument [?, ?], while recent Isabelle/HOL work on verified rewriting infrastructure for diagrammatic equivalence [?] offers another benchmark for future formalisation.



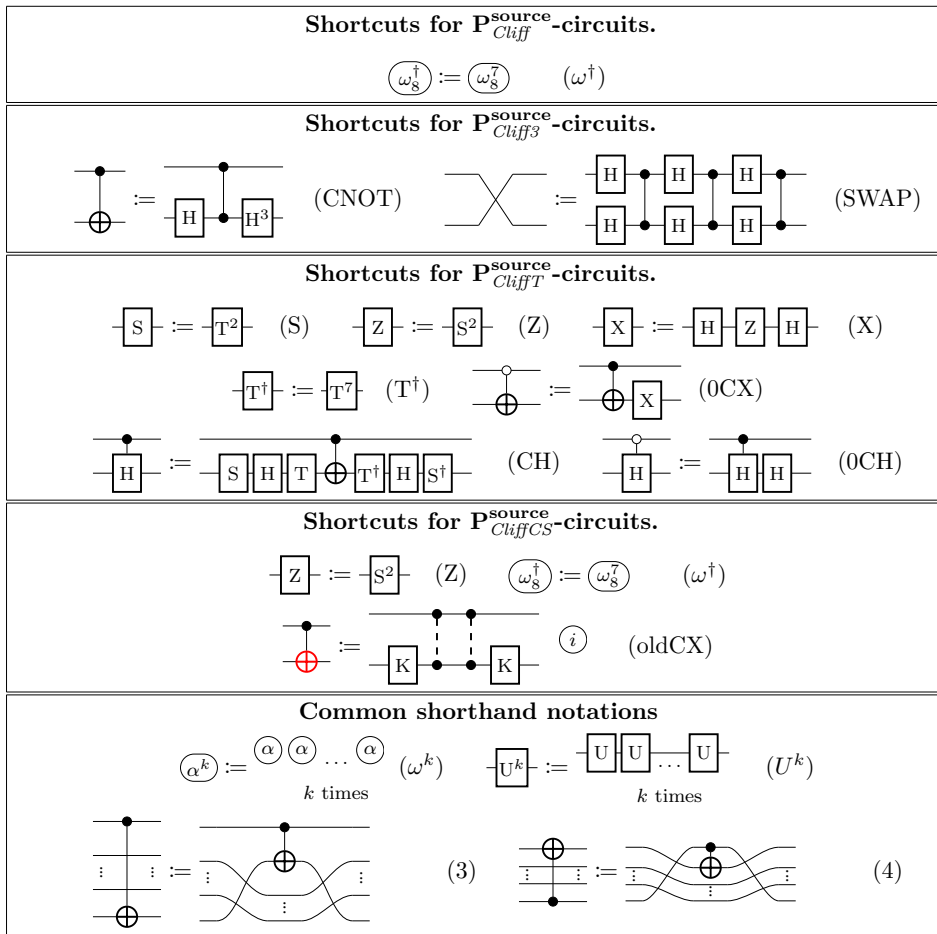
■ **Figure 9** Summary of Shortcuts used for all considered fragments.

A

 Auxiliary derivations for Cliff_2

Derivation A.1.

$$\boxed{Z} \boxed{Z} \stackrel{(Z)}{=} \boxed{S} \boxed{S} \boxed{S} \boxed{S} \stackrel{(S^4)}{=} \text{---}$$



■ **Figure 10** Summary of Shortcuts used for all considered fragments used in the original PROs.

Derivation A.2.

$$-X-X \stackrel{(X)}{=} -H-Z-H-H-Z-H \stackrel{(H^2)}{=} -H-Z-Z-H \stackrel{(A.1)}{=} -H-H \stackrel{(H^2)}{=} \text{---}$$

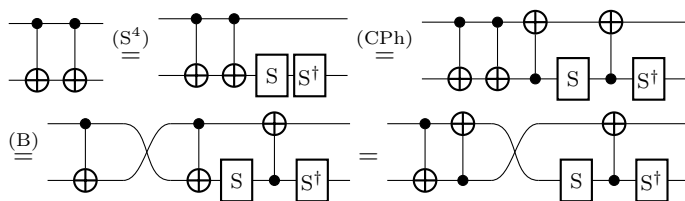
Derivation A.3.

$$\begin{aligned} -X-S-X &\stackrel{(X)}{=} -H-S-S-H-S-H-S-S-H \\ &\stackrel{(E)}{=} \omega_8 -H-S-S-S^\dagger-H-S^\dagger-S-S-H \stackrel{(S^4)}{=} \omega_8 -H-S-H-S-H \\ &\stackrel{(E)}{=} \omega_8^2 -S^\dagger-H-S^\dagger-S-H \stackrel{(S^4)(H^2)}{=} \omega_8^2 -S^\dagger \end{aligned}$$

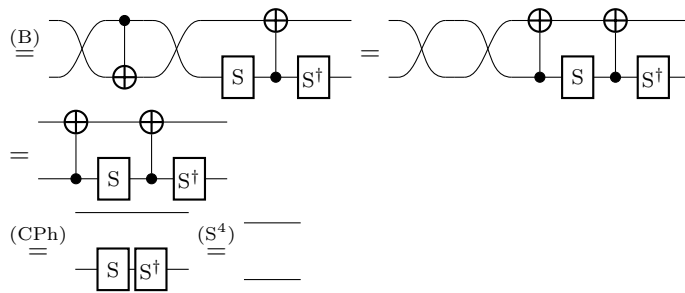
Derivation A.4.

$$-H \stackrel{(S^4)}{=} -H-S^\dagger-S \stackrel{(H^2)}{=} -H-S^\dagger-H-H-S \stackrel{(E)}{=} \omega_8^\dagger -S-H-S-H-S$$

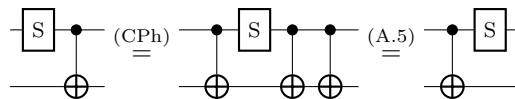
Derivation A.5.



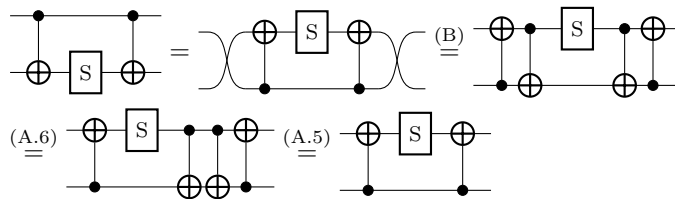
3:20 **Simpler Presentations for Many Fragments of Quantum Circuits**



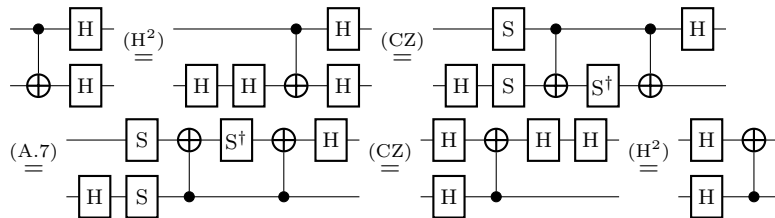
Derivation A.6.



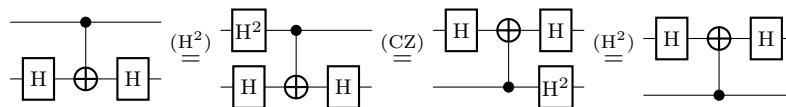
Derivation A.7.



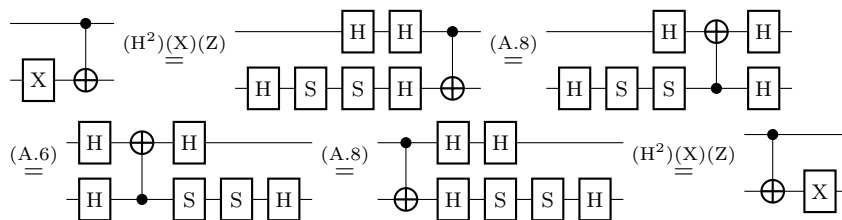
Derivation A.8.



Derivation A.9.

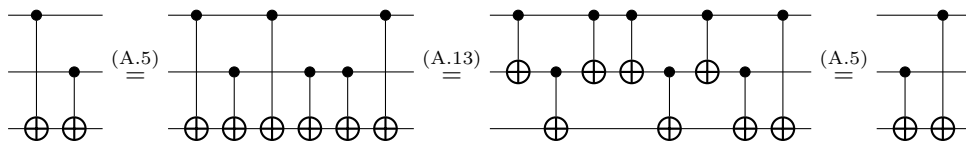


Derivation A.10.

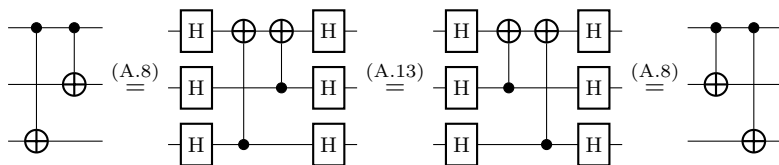


3:22 Simpler Presentations for Many Fragments of Quantum Circuits

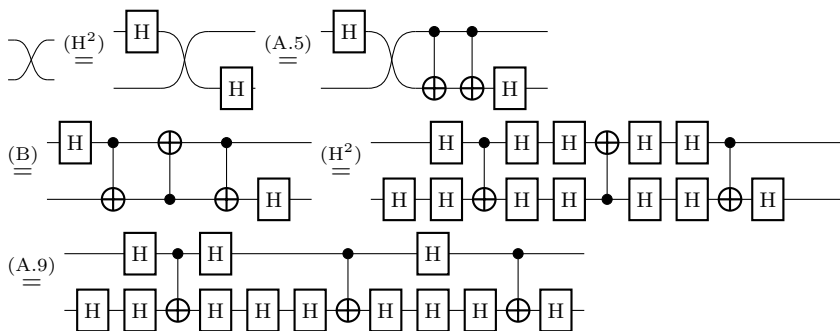
Derivation A.14.



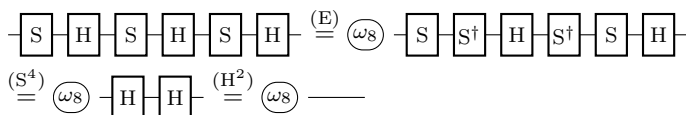
Derivation A.15.



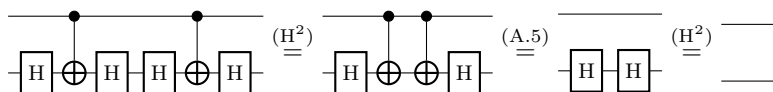
Derivation A.16.



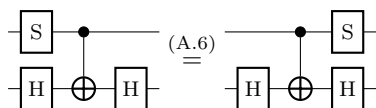
Derivation A.17.



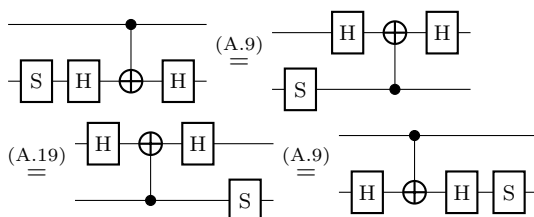
Derivation A.18.



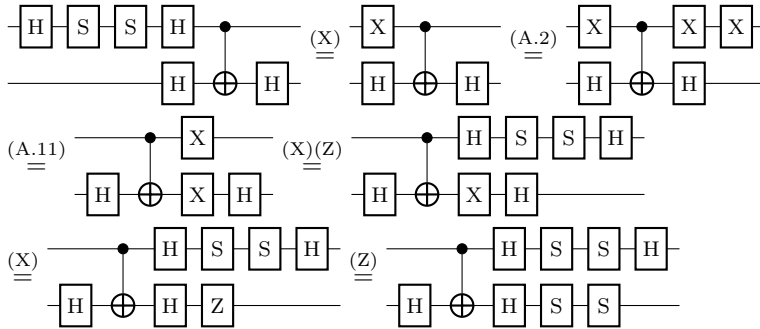
Derivation A.19.



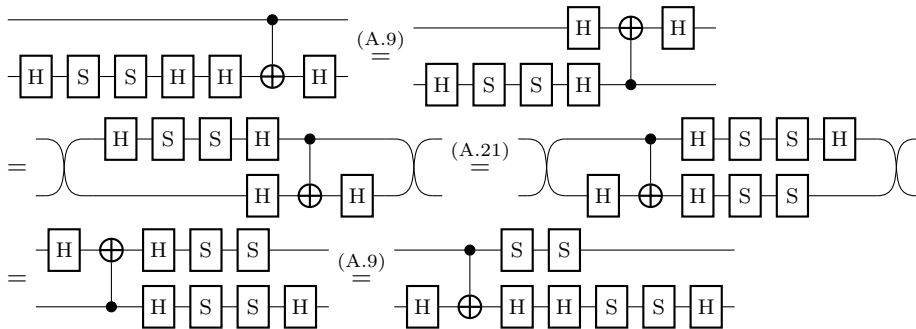
Derivation A.20.



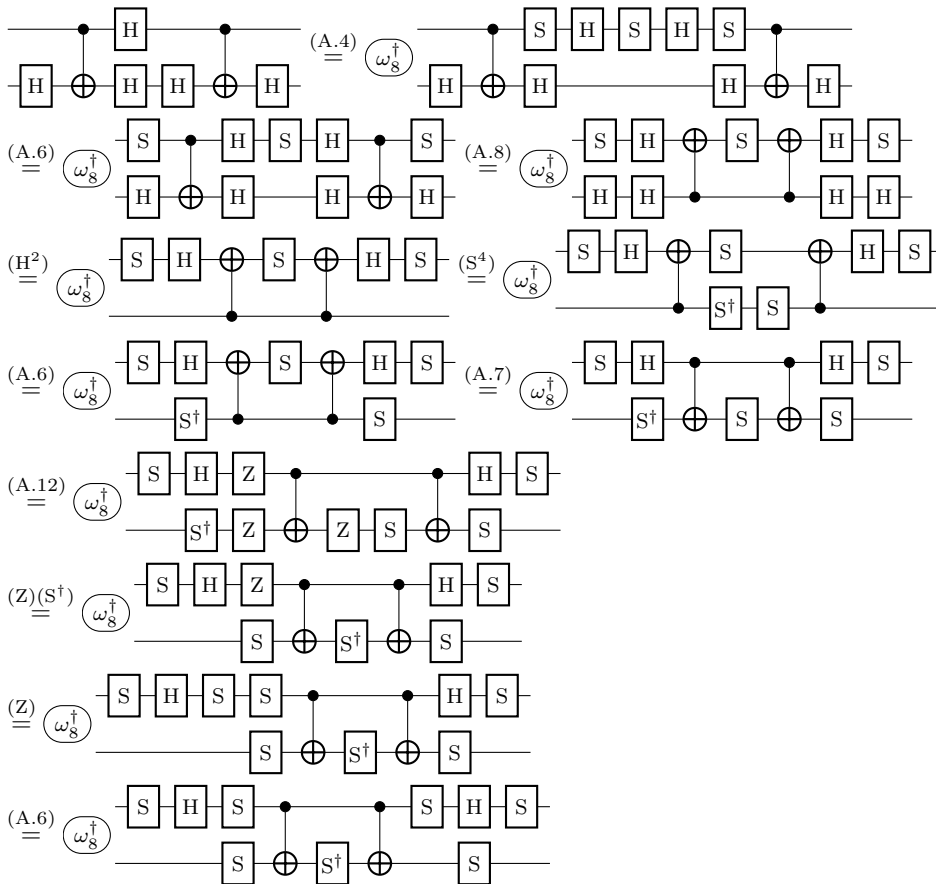
Derivation A.21.

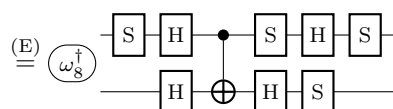


Derivation A.22.

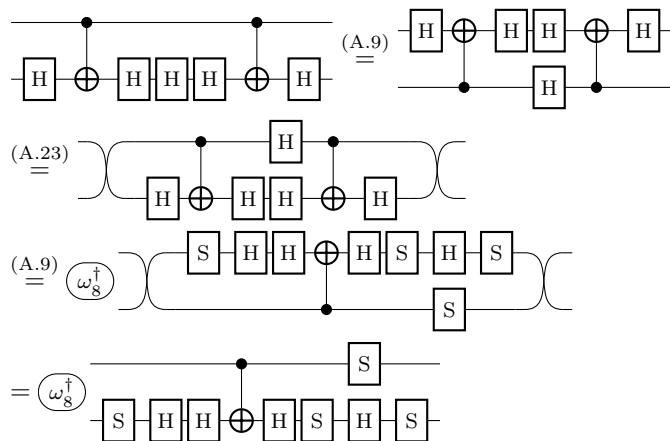


Derivation A.23.

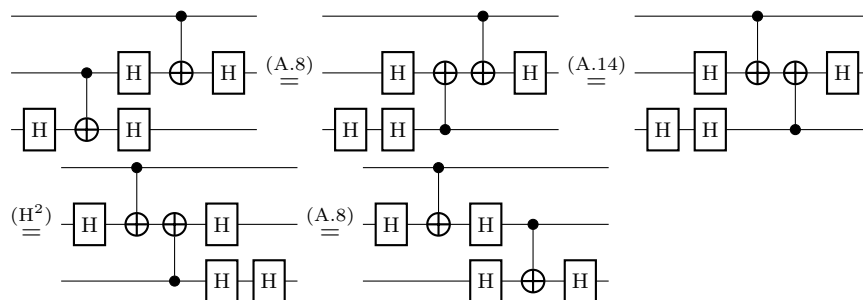




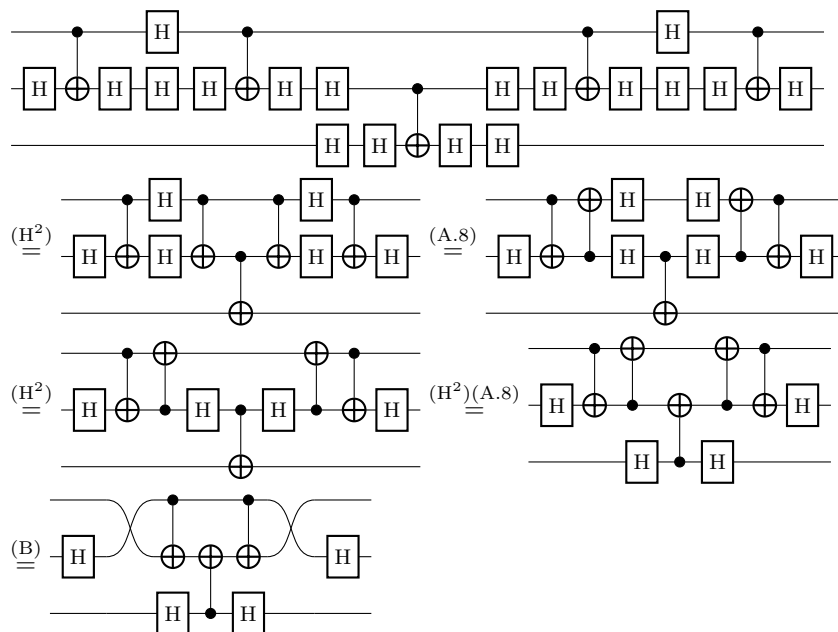
Derivation A.24.

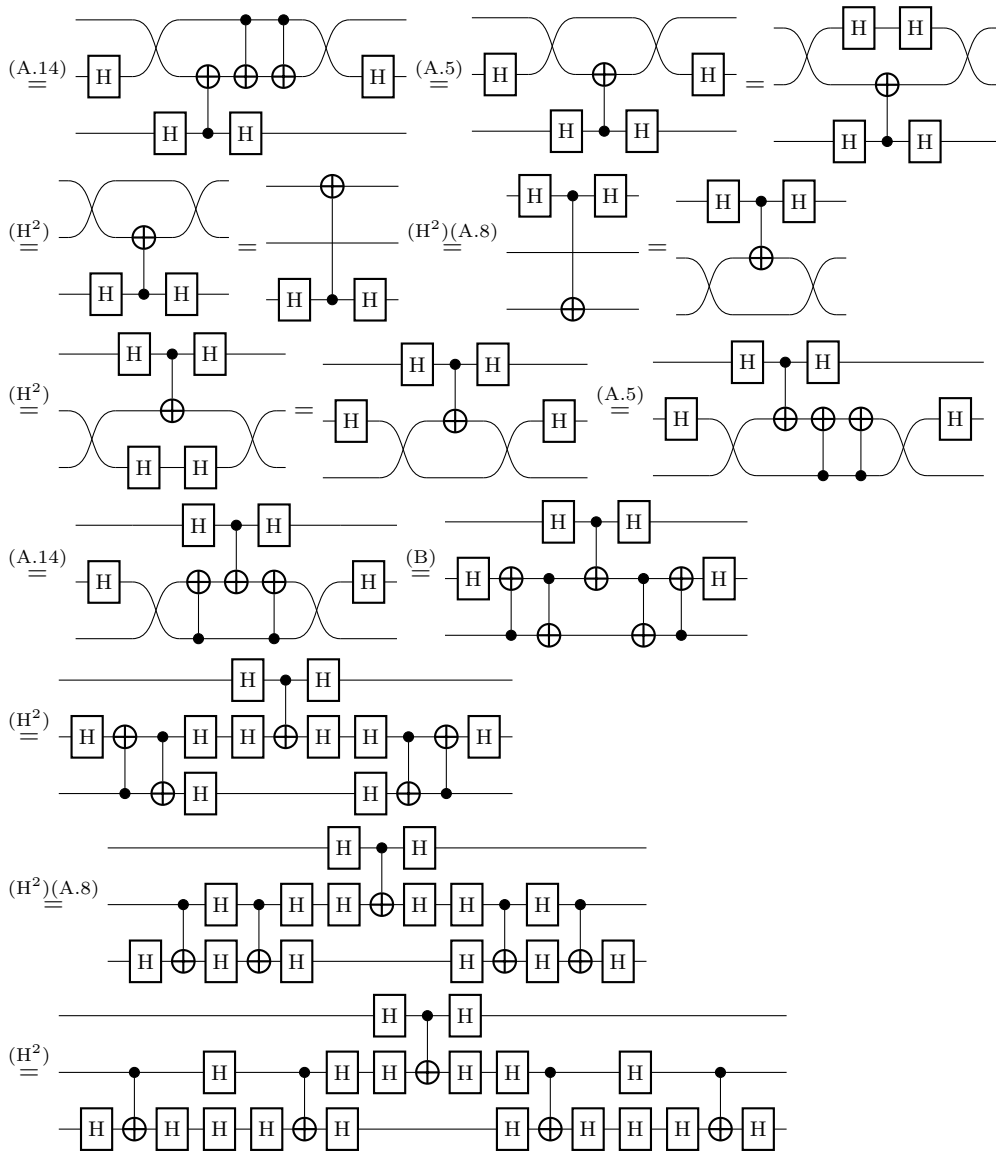


Derivation A.25.

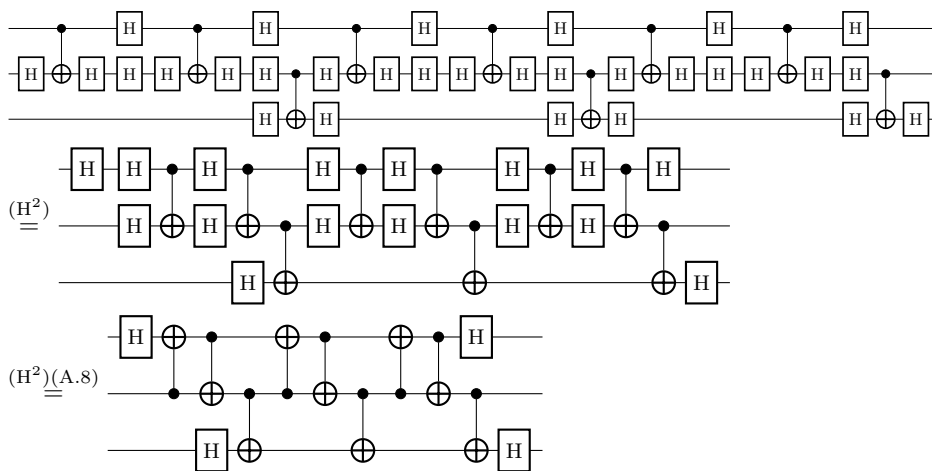


Derivation A.26.

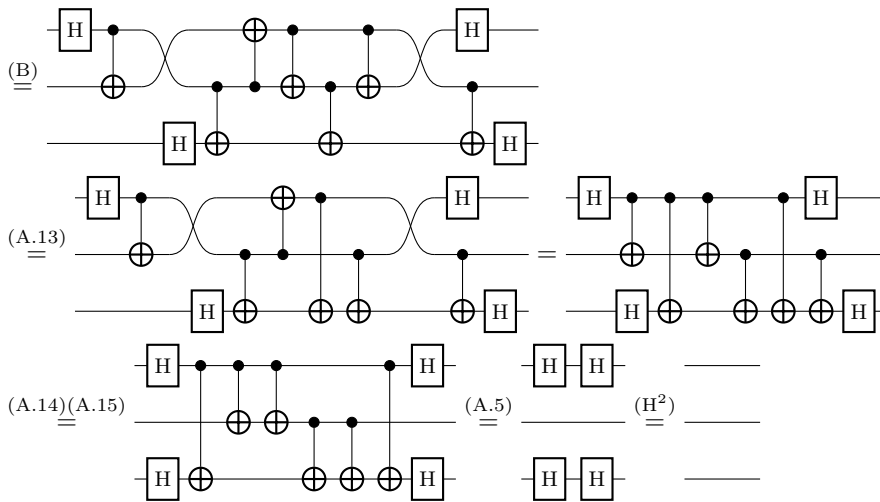




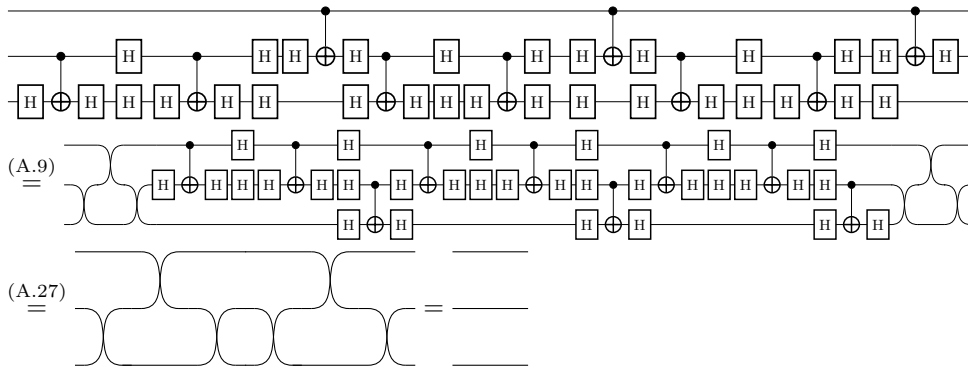
Derivation A.27.



3:26 **Simpler Presentations for Many Fragments of Quantum Circuits**

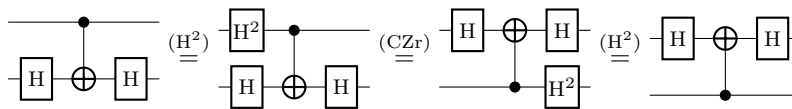


Derivation A.28.

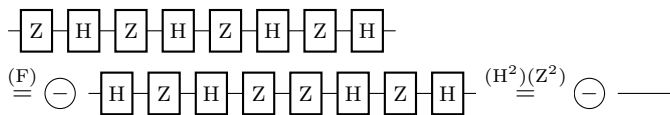


B **Auxiliary derivations for RCliff₂**

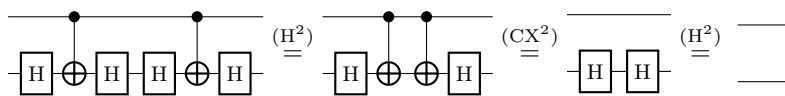
Derivation B.1.



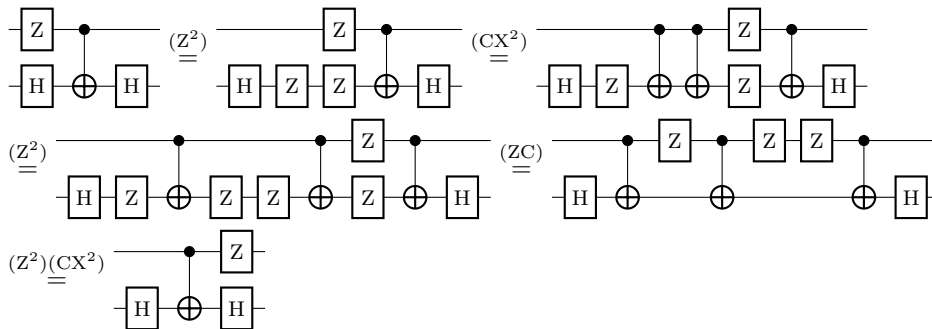
Derivation B.2.



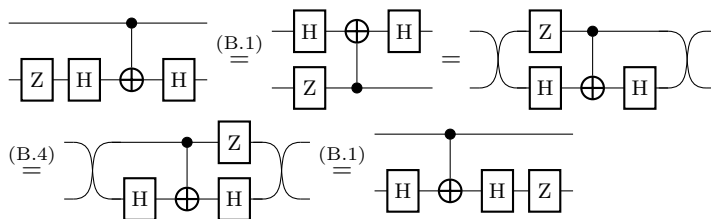
Derivation B.3.



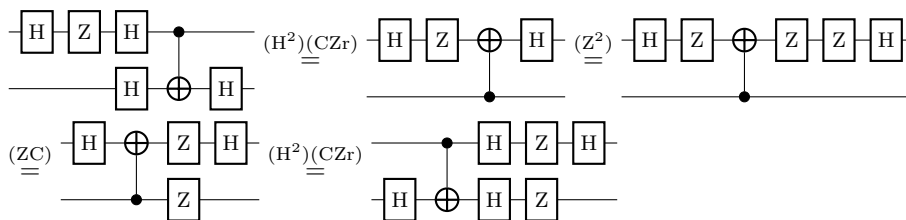
Derivation B.4.



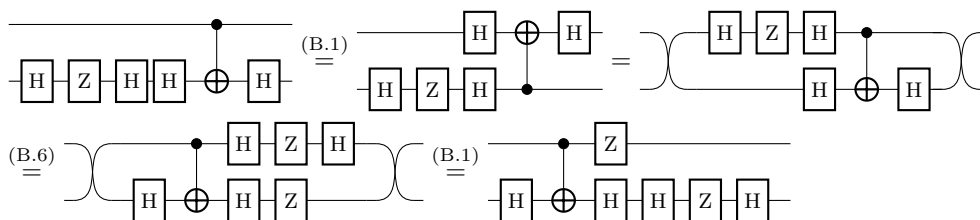
Derivation B.5.



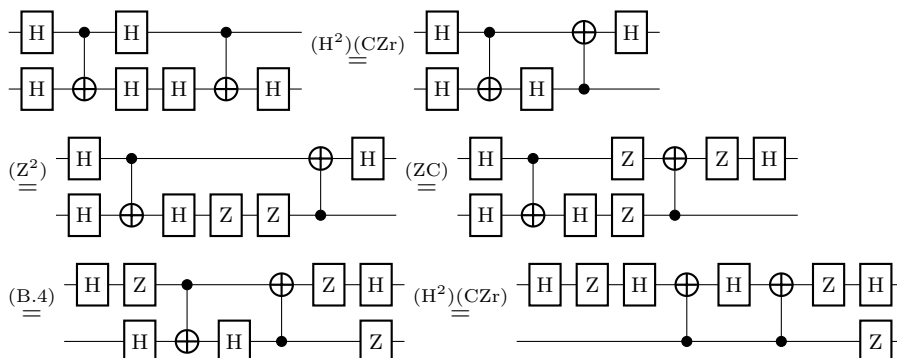
Derivation B.6.

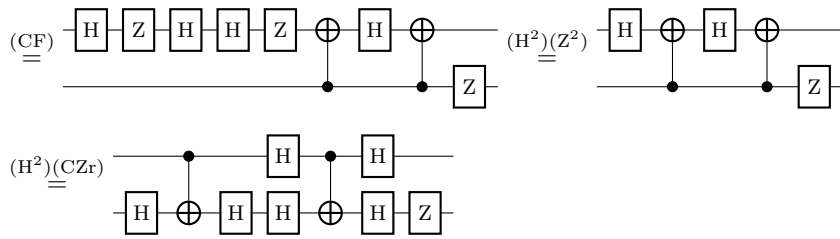


Derivation B.7.

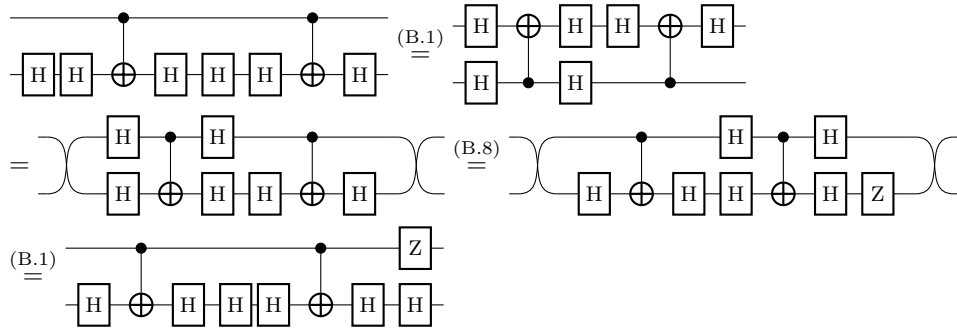


Derivation B.8.

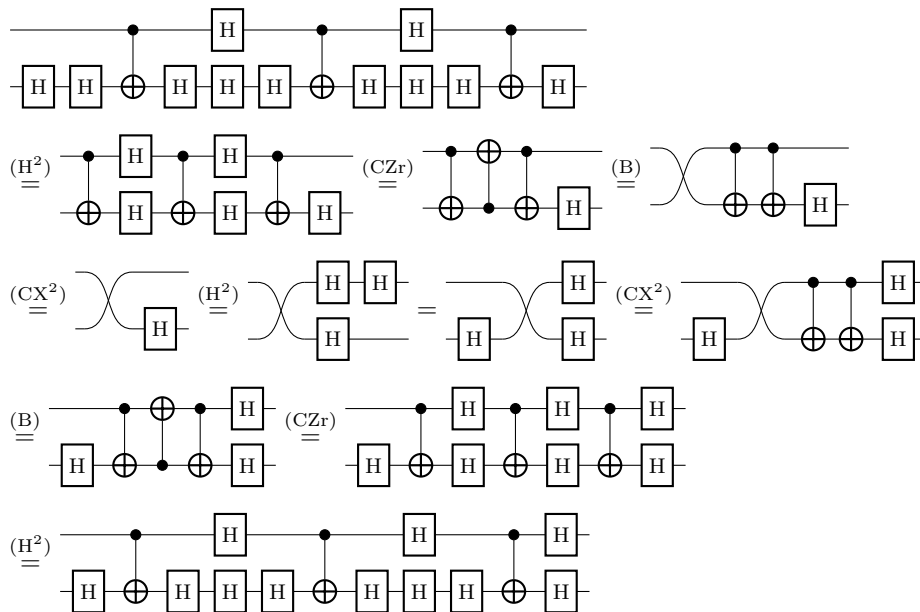




Derivation B.9.

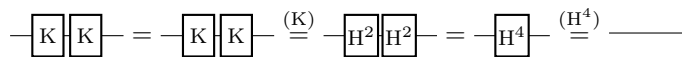


Derivation B.10.

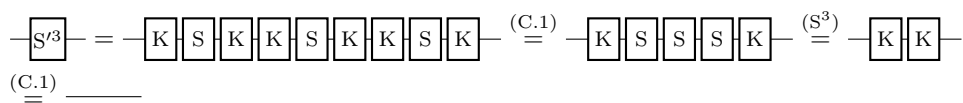


C Auxiliary derivations for Cliff_3

Derivation C.1.



Derivation C.2.



Derivation C.3.

$$\underline{Z^3} \stackrel{(Z)}{=} \underline{S \ S'2 \ S \ S'2 \ S \ S'2} \stackrel{(SS')}{=} \underline{S^3 \ S'6} \stackrel{(S^3)(C.2)}{=} \underline{\quad}$$

Derivation C.4.

$$\underline{X^3} \stackrel{(X)}{=} \underline{H \ Z \ H^3 \ H \ Z \ H^3 \ H \ Z \ H^3} \stackrel{(H^4)}{=} \underline{H \ Z \ Z \ Z \ H^3} \\ \stackrel{(C.3)}{=} \underline{H \ H^3} \stackrel{(H^4)}{=} \underline{\quad}$$

Derivation C.5.

$$\underline{S \ K} \stackrel{(C.1)}{=} \underline{K \ K \ S \ K} \stackrel{(S')}{=} \underline{K \ S'}$$

Derivation C.6.

$$\underline{S' \ K} \stackrel{(S')}{=} \underline{K \ S \ K \ K} \stackrel{(C.1)}{=} \underline{K \ S}$$

Derivation C.7.

$$\underline{Z \ K} \stackrel{(Z)}{=} \underline{S \ S'2 \ K} \stackrel{(C.6)}{=} \underline{S \ K \ S^2} \stackrel{(C.5)}{=} \underline{K \ S' \ S^2} \\ \stackrel{(C.2)}{=} \underline{K \ S'2 \ S'2 \ S \ S} \stackrel{(SS')}{=} \underline{K \ S \ S'2 \ S \ S'2} \stackrel{(Z)}{=} \underline{K \ Z \ Z} = \underline{K \ Z^2}$$

Derivation C.8.

$$\underline{X \ K} \stackrel{(X)}{=} \underline{H \ Z \ H^\dagger \ K} \stackrel{(K)}{=} \underline{H \ Z \ K \ H^\dagger} \stackrel{(C.7)}{=} \underline{K \ H \ Z^2 \ H^\dagger} \\ \stackrel{(K)}{=} \underline{K \ H \ Z^2 \ H^\dagger} \stackrel{(H^4)(X)}{=} \underline{K \ X^2}$$

Derivation C.9.

$$\underline{X \ H} \stackrel{(X)}{=} \underline{H \ Z \ H^3 \ H} \stackrel{(H^4)}{=} \underline{H \ Z}$$

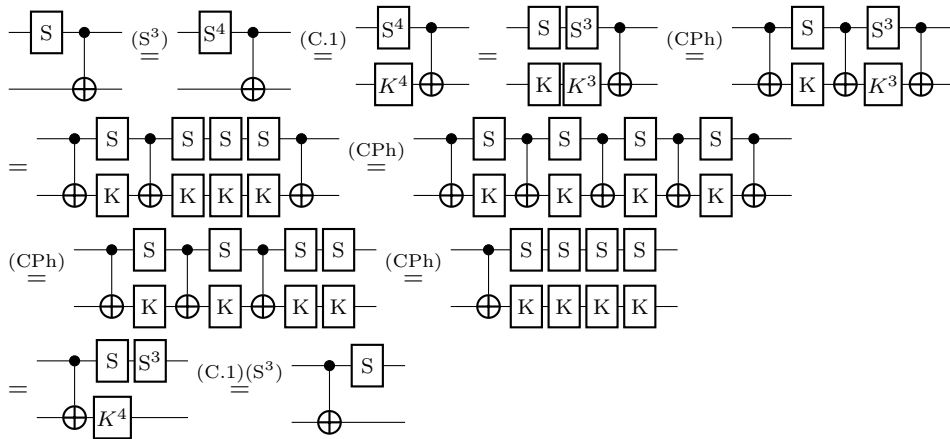
Derivation C.10.

$$\underline{Z^2 \ H} \stackrel{(H^4)(K)}{=} \underline{Z^2 \ K \ H^3} \stackrel{(C.7)(C.3)}{=} \underline{K \ Z \ H^3} \\ \stackrel{(K)}{=} \underline{H \ H \ Z \ H^3} \stackrel{(X)}{=} \underline{H \ X}$$

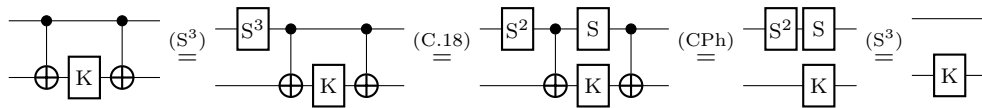
Derivation C.11.

$$\underline{X \ S} \stackrel{(X)}{=} \underline{H \ Z \ H^\dagger \ S} \stackrel{(Z)}{=} \underline{H \ S^2 \ S' \ H^\dagger \ S} \stackrel{(H^4)(S')}{=} \underline{H \ S^2 \ H^2 \ S \ H \ S} \\ \stackrel{(E)}{=} \underline{H \ S^2 \ H^2 \ H^3 \ S^2 \ H^3} \stackrel{(\omega_{12}^7)}{=} \underline{H^2 \ H^3 \ S^2 \ H^3 \ H^2 \ S^2 \ H^3} \\ \stackrel{(E)}{=} \underline{H^2 \ S \ H \ S \ H^2 \ S^2 \ H^3} \stackrel{(C.5)(S')}{=} \underline{H^2 \ S \ H^3 \ S' \ S^2 \ H^3} \\ \stackrel{(E)(H^4)(S')}{=} \underline{S' \ H \ S' \ S^2 \ H^3} \stackrel{(Z)(X)}{=} \underline{S' \ X} \\ \stackrel{(Z)(S^3)(C.2)}{=} \underline{Z \ S \ X} \stackrel{(SS')}{=} \underline{S \ Z \ X}$$

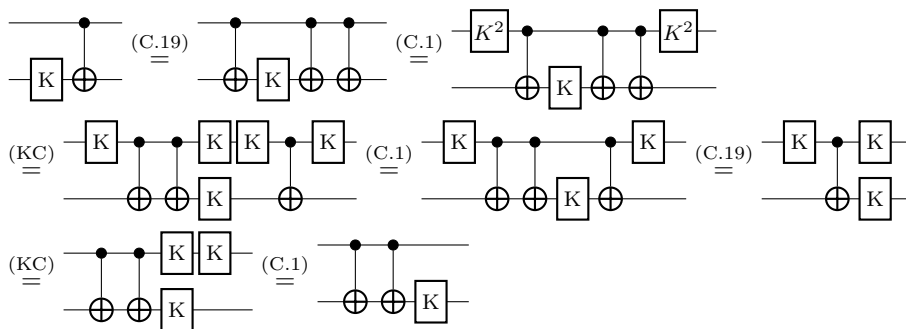
Derivation C.18.



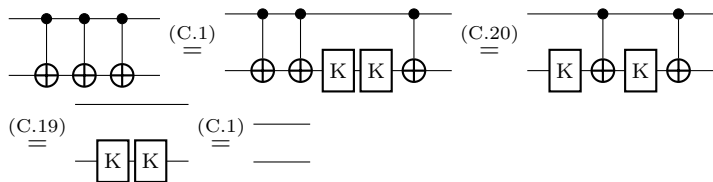
Derivation C.19.



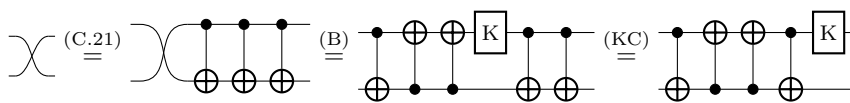
Derivation C.20.



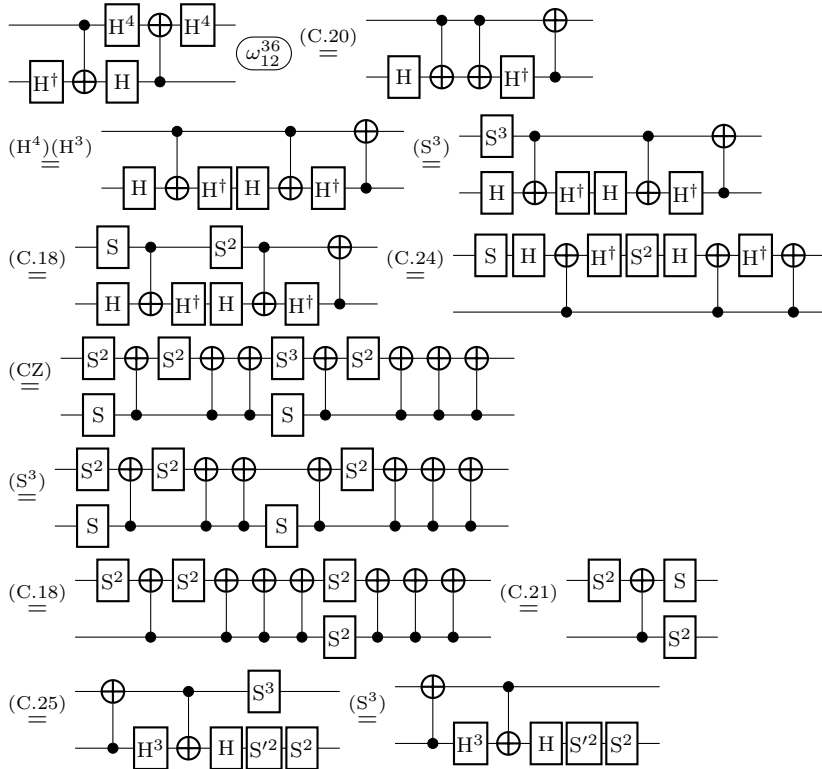
Derivation C.21.



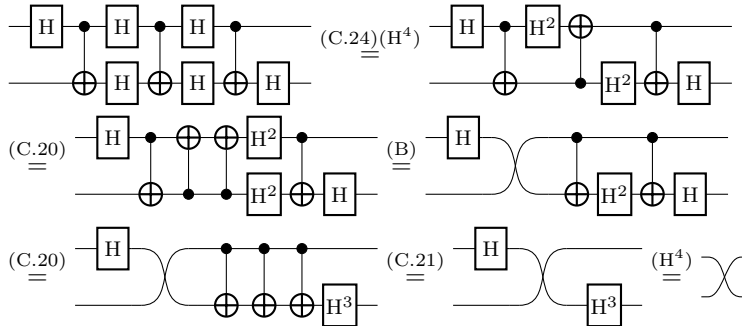
Derivation C.22.



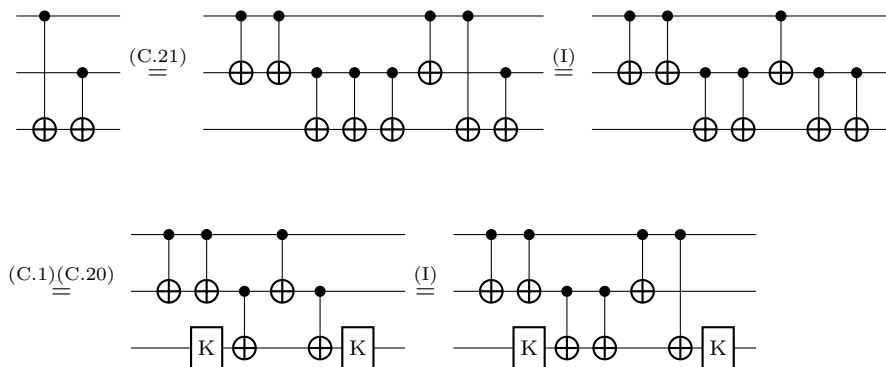
Derivation C.26.



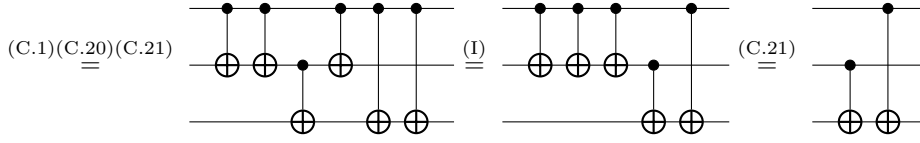
Derivation C.27.



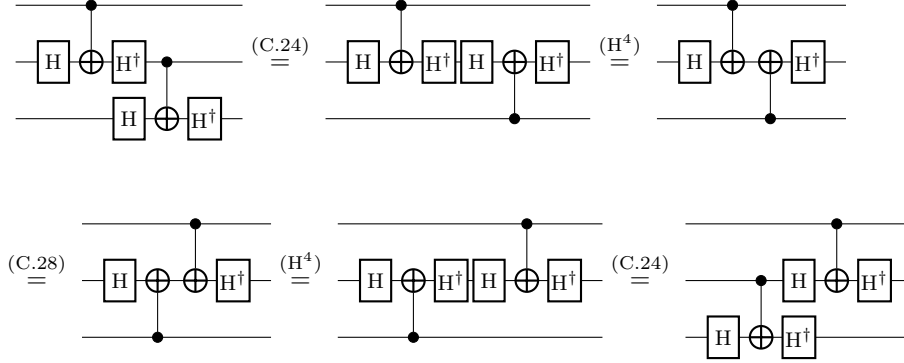
Derivation C.28.



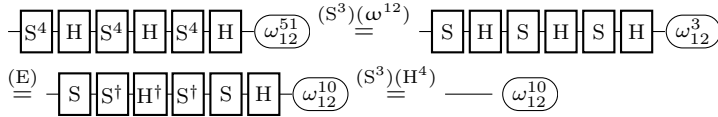
3:34 Simpler Presentations for Many Fragments of Quantum Circuits



Derivation C.29.



Derivation C.30.



► **Proposition 38.** *The axiom (SS') is necessary in Cliff₃.*

Proof. The interpretation $\llbracket \cdot \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket}$ respects all axioms of Cliff₃ acting on at most one qutrit except Equation (SS'). The checks are not merely syntactic: for instance, (S³) and (E) remain sound because the relevant images reduce as follows.

$$\llbracket \llbracket -S \llbracket S \llbracket S \llbracket - \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} = \llbracket \llbracket -S \llbracket X \llbracket S \llbracket X \llbracket S \llbracket X \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \stackrel{(C.14)}{=} \llbracket \llbracket - \llbracket \omega_{12}^4 \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \sim \llbracket \llbracket - \llbracket \omega_{12}^4 \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket}$$

$$\llbracket \llbracket -S \llbracket H \llbracket S \llbracket - \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} = \llbracket \llbracket -S \llbracket X \llbracket H \llbracket S \llbracket X \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \stackrel{(C.17)}{=} \llbracket \llbracket -H^3 \llbracket S \llbracket X \llbracket S \llbracket X \llbracket H^3 \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \llbracket \llbracket \omega_{12}^3 \rrbracket \rrbracket \sim \llbracket \llbracket -H \llbracket H \llbracket H \llbracket S \llbracket S \llbracket H \llbracket H \llbracket H \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \llbracket \llbracket \omega_{12}^7 \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket}$$

The axiom (SS') is the point where the interpretation separates the theory. Its two sides are sent to inequivalent elements of Qudit₃[~](1, 1):

$$\llbracket \llbracket -S \llbracket K \llbracket S \llbracket K \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} = \llbracket \llbracket -S \llbracket X \llbracket K \llbracket S \llbracket X \llbracket K \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \stackrel{(C.15)}{=} \llbracket \llbracket -S^2 \llbracket \omega_{12}^4 \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket},$$

$$\llbracket \llbracket -K \llbracket S \llbracket K \llbracket S \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} = \llbracket \llbracket -K \llbracket S \llbracket X \llbracket K \llbracket S \llbracket X \llbracket - \rrbracket \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket} \stackrel{(C.16)}{=} \llbracket \llbracket -S^2 \llbracket \omega_{12}^4 \rrbracket \rrbracket \rrbracket_{Cliff3, \sim}^{\llbracket \cdot \rrbracket := \llbracket \cdot \rrbracket - \llbracket \cdot \rrbracket}.$$

All other axioms acting on at most one qutrit remain sound under this interpretation. Removing Equation (SS') would therefore identify too little and leave the resulting theory incomplete. ◀

D Auxiliary derivations for CliffT₂

Derivation D.1.

$$\boxed{Z} \boxed{Z} \stackrel{(Z)}{=} \boxed{S} \boxed{S} \boxed{S} \boxed{S} \stackrel{(S)}{=} \boxed{T^8} \stackrel{(T^8)}{=} \text{---}$$

Derivation D.2.

$$\boxed{X} \boxed{X} \stackrel{(X)}{=} \boxed{H} \boxed{Z} \boxed{H} \boxed{H} \boxed{Z} \boxed{H} \stackrel{(H^2)}{=} \boxed{H} \boxed{Z} \boxed{Z} \boxed{H} \stackrel{(D.1)}{=} \boxed{H} \boxed{H} \stackrel{(H^2)}{=} \text{---}$$

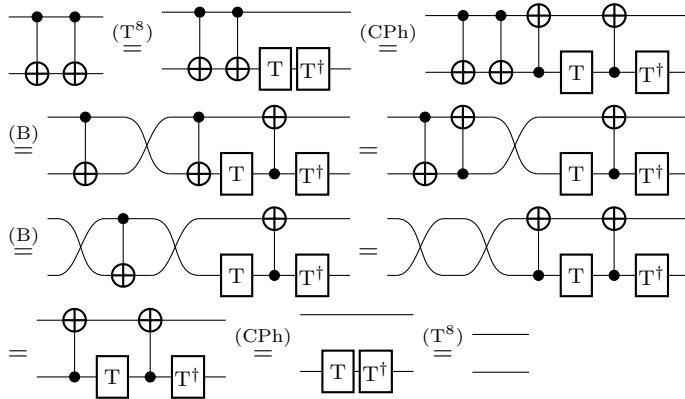
Derivation D.3.

$$\boxed{H} \boxed{X} \stackrel{(X)}{=} \boxed{H} \boxed{H} \boxed{Z} \boxed{H} \stackrel{(H^2)}{=} \boxed{Z} \boxed{H}$$

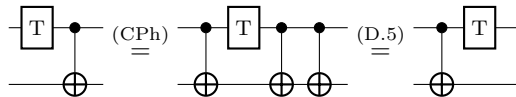
Derivation D.4.

$$\begin{aligned} & \boxed{X} \boxed{S} \boxed{X} \stackrel{(X)}{=} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \\ & \stackrel{(E)}{=} (\omega_8) \boxed{H} \boxed{S} \boxed{S} \boxed{S^\dagger} \boxed{H} \boxed{S^\dagger} \boxed{S} \boxed{S} \boxed{H} \stackrel{(T^8)}{=} (\omega_8) \boxed{H} \boxed{S} \boxed{H} \boxed{S} \boxed{H} \\ & \stackrel{(E)}{=} (\omega_8^2) \boxed{S^\dagger} \boxed{H} \boxed{S^\dagger} \boxed{S} \boxed{H} \stackrel{(T^8)(H^2)}{=} (\omega_8^2) \boxed{S^\dagger} \end{aligned}$$

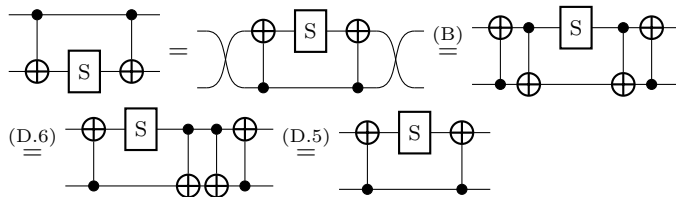
Derivation D.5.



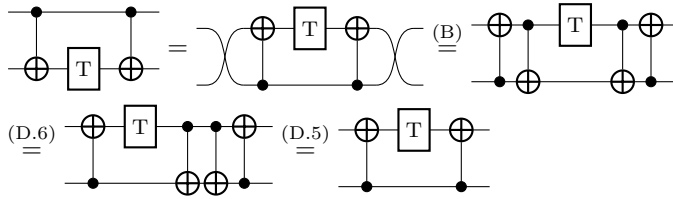
Derivation D.6.



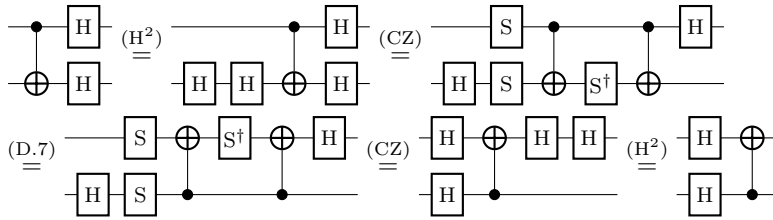
Derivation D.7.



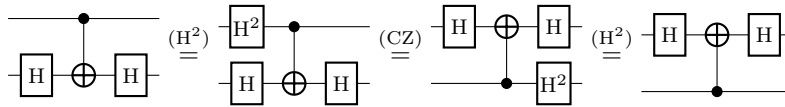
Derivation D.8.



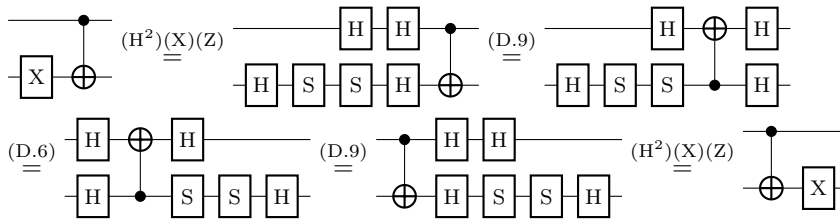
Derivation D.9.



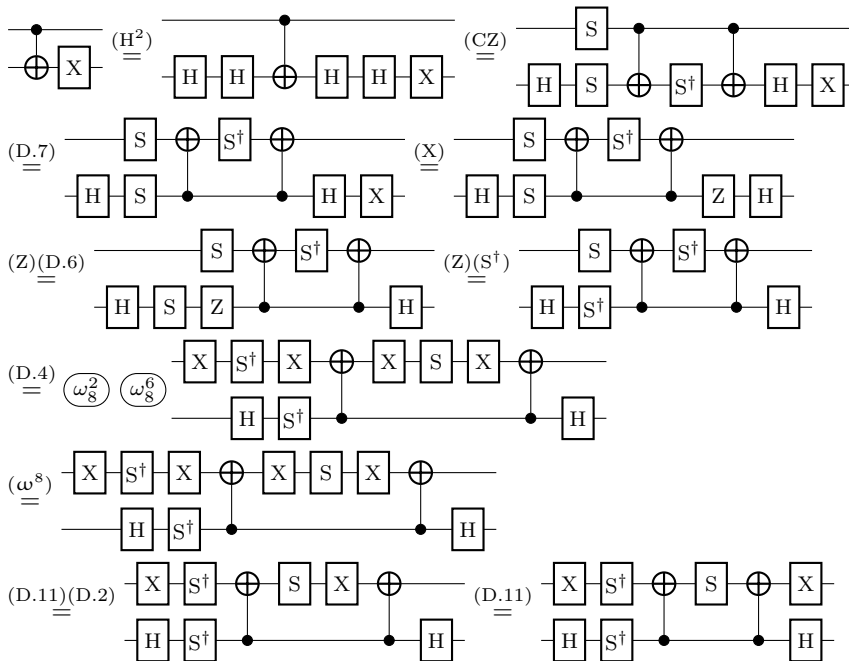
Derivation D.10.

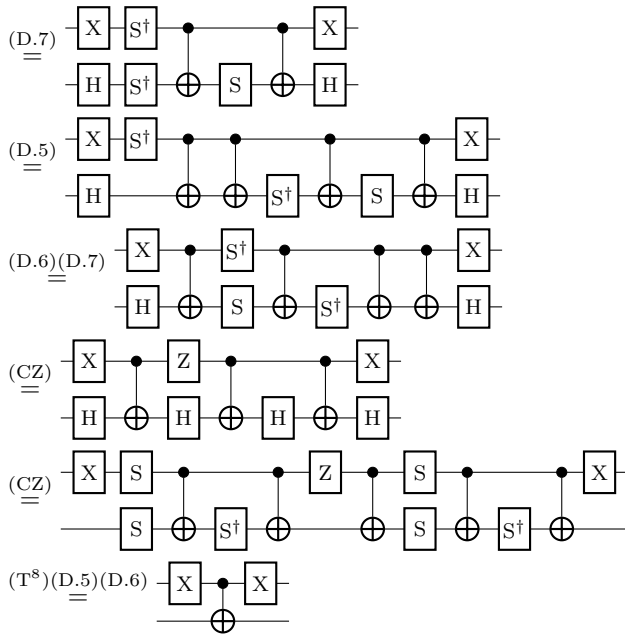


Derivation D.11.

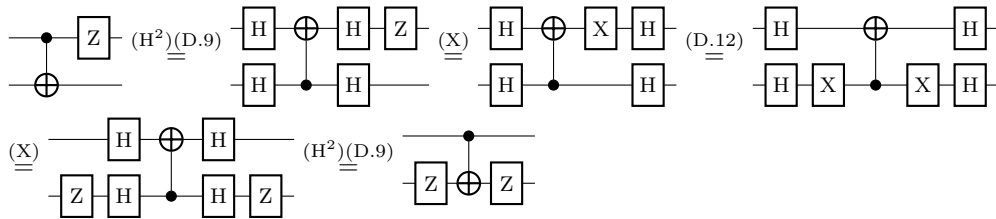


Derivation D.12.

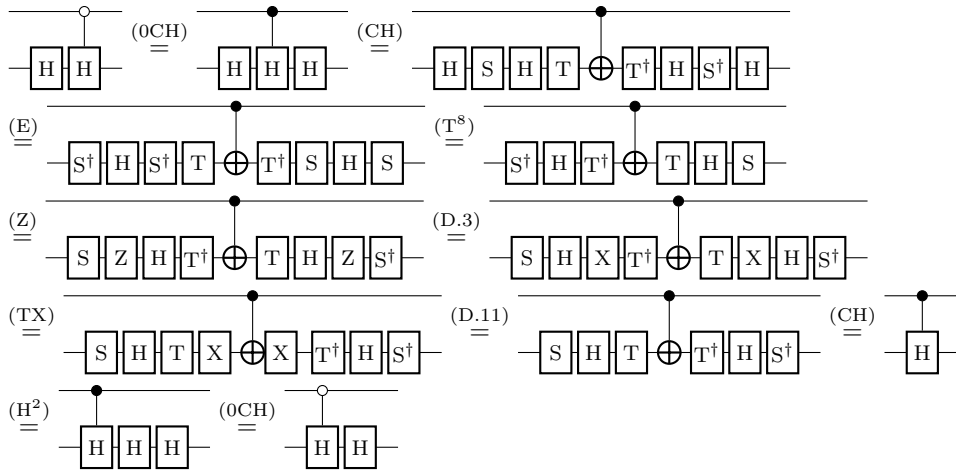




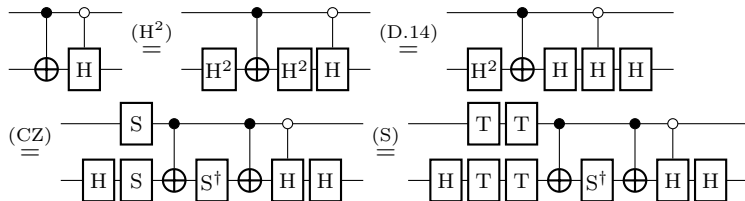
Derivation D.13.



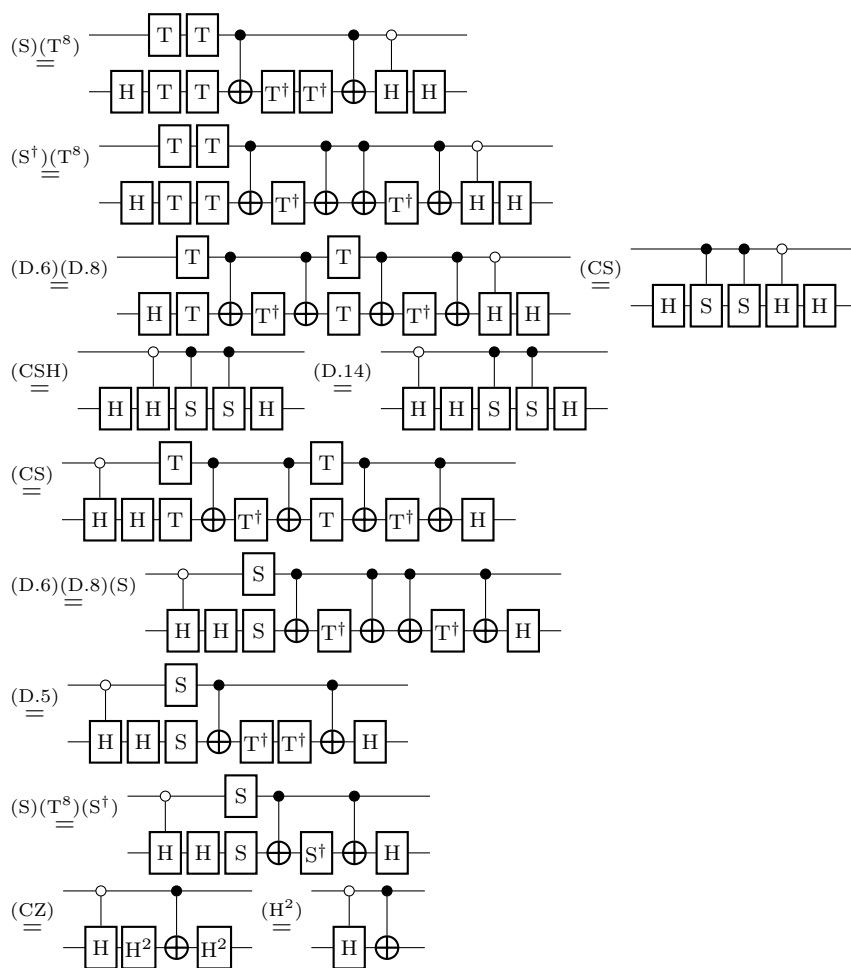
Derivation D.14.



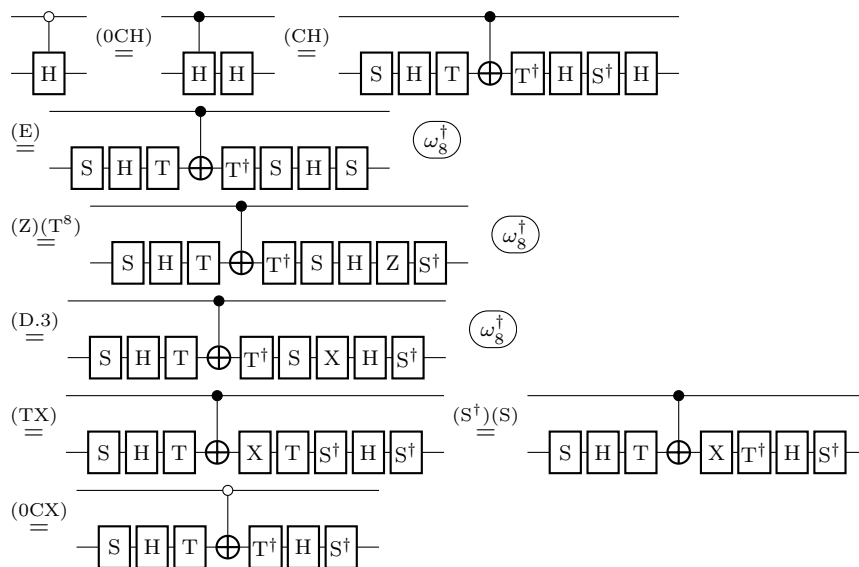
Derivation D.15.



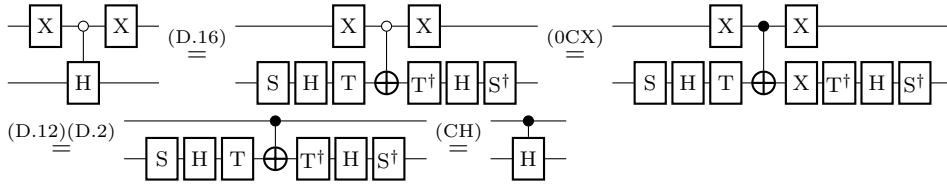
3:38 Simpler Presentations for Many Fragments of Quantum Circuits



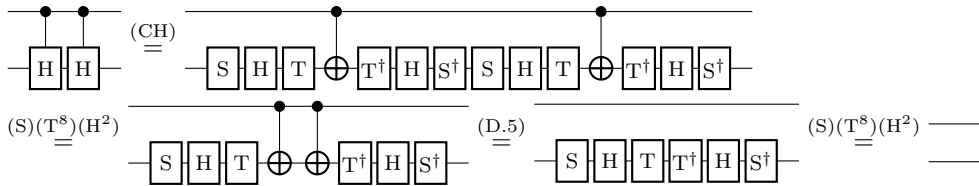
Derivation D.16.



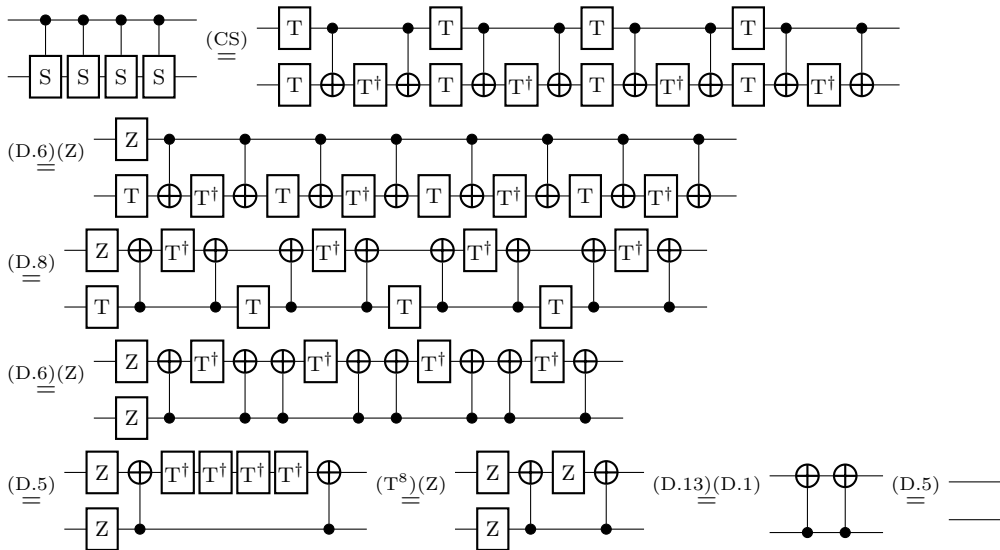
Derivation D.17.



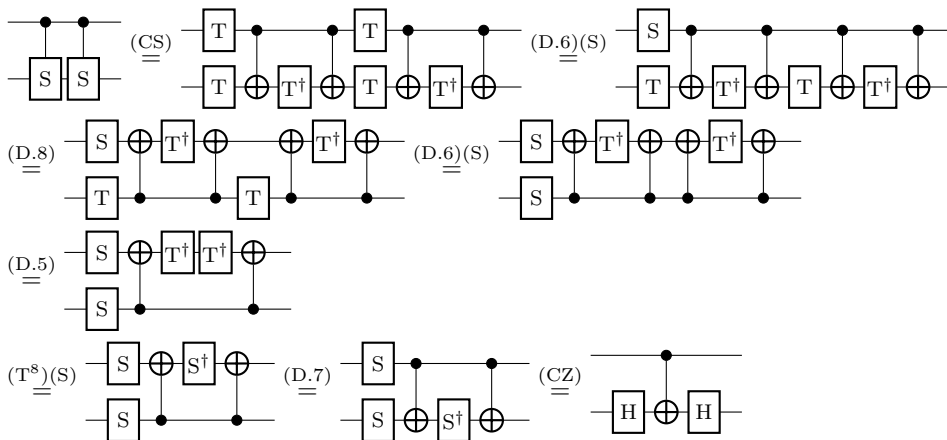
Derivation D.18.



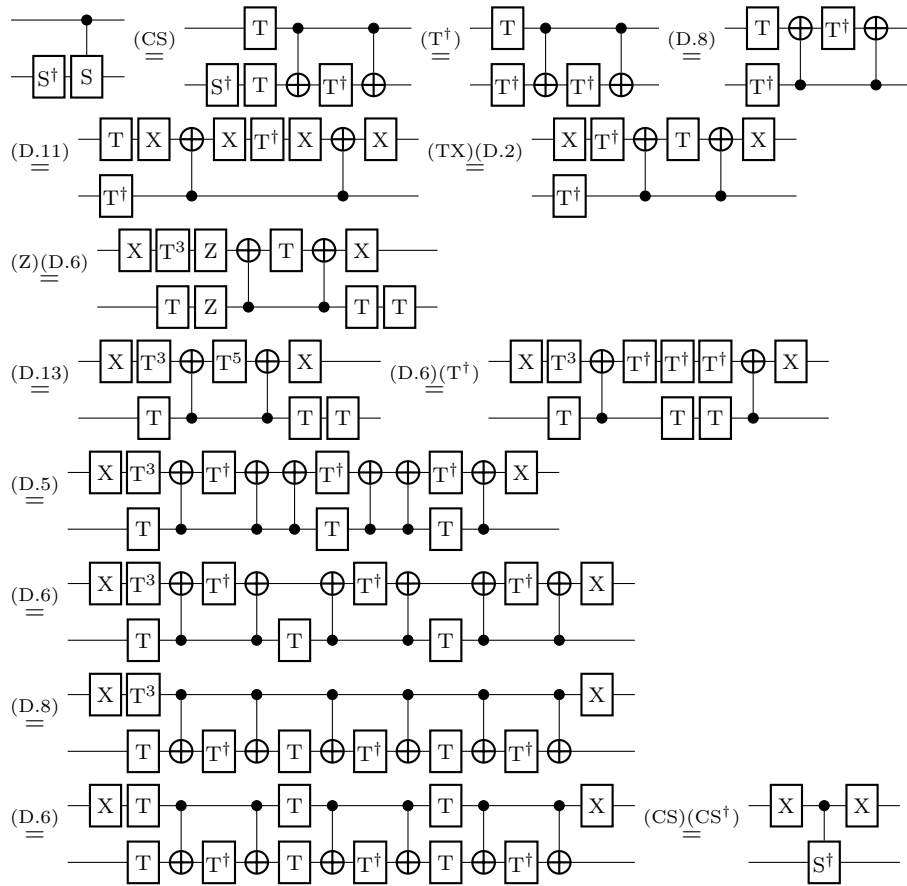
Derivation D.19.



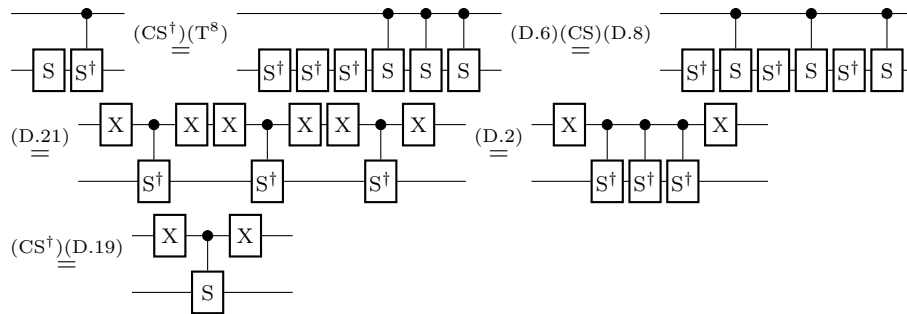
Derivation D.20.



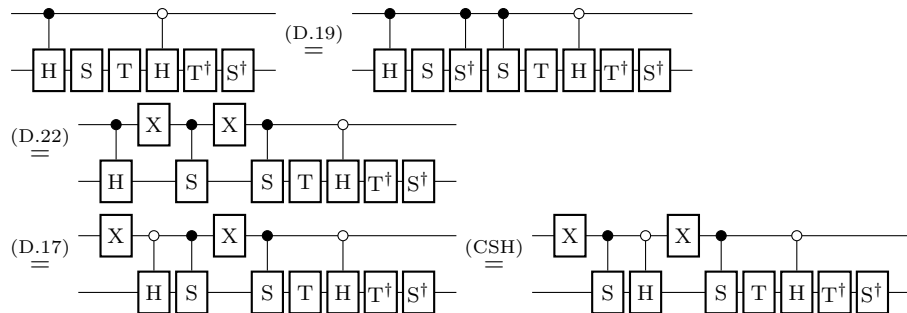
Derivation D.21.

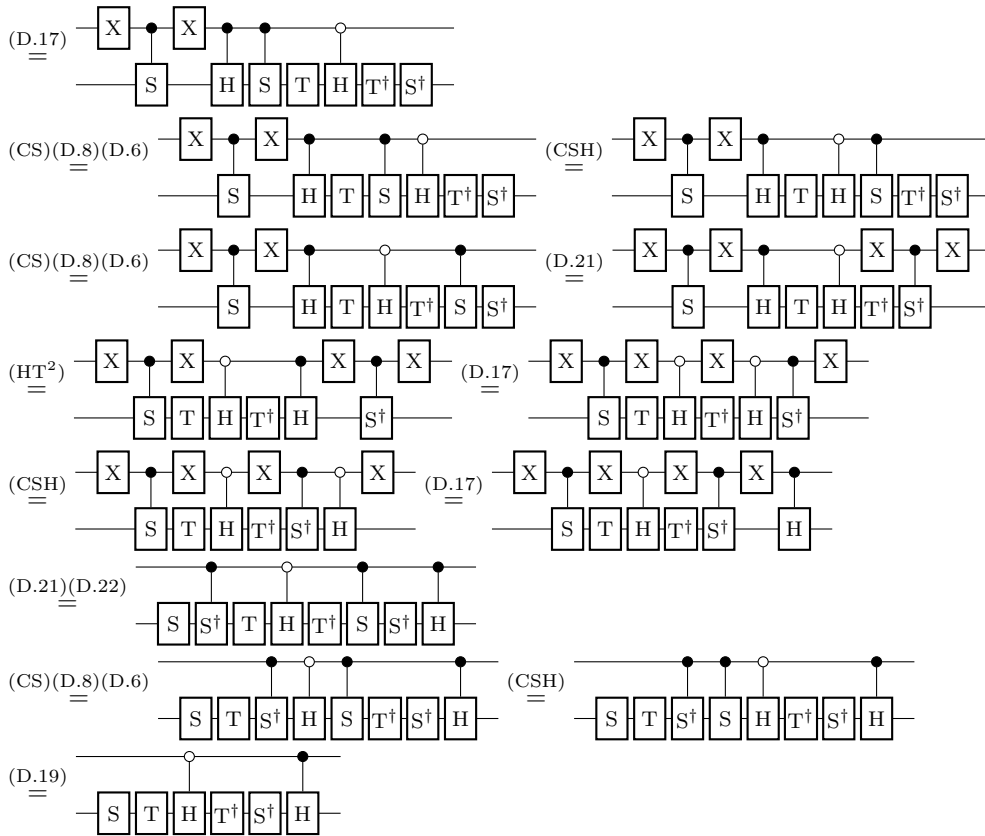


Derivation D.22.

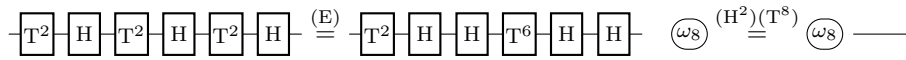


Derivation D.23.

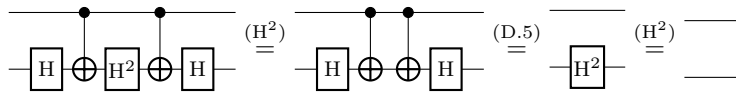




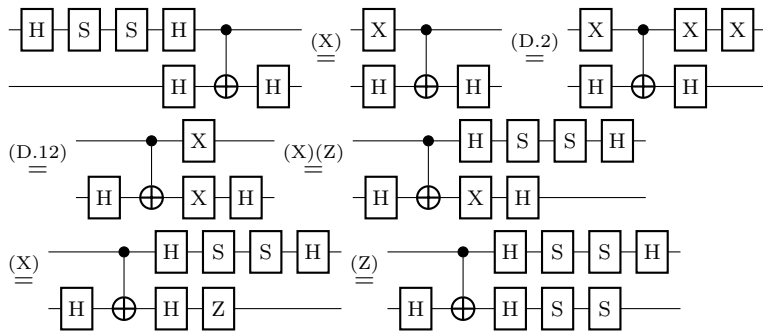
Derivation D.24.

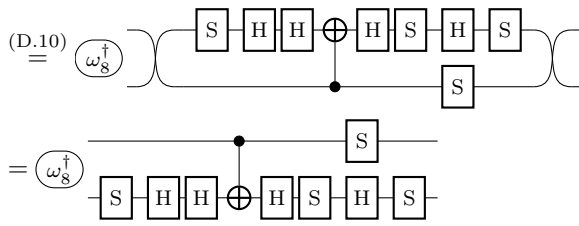


Derivation D.25.

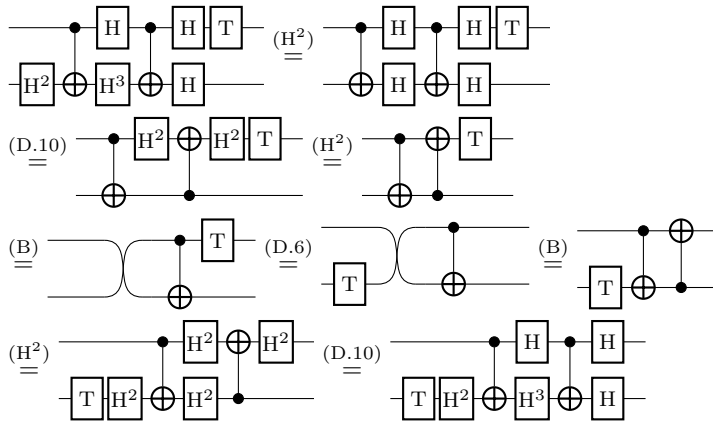


Derivation D.26.

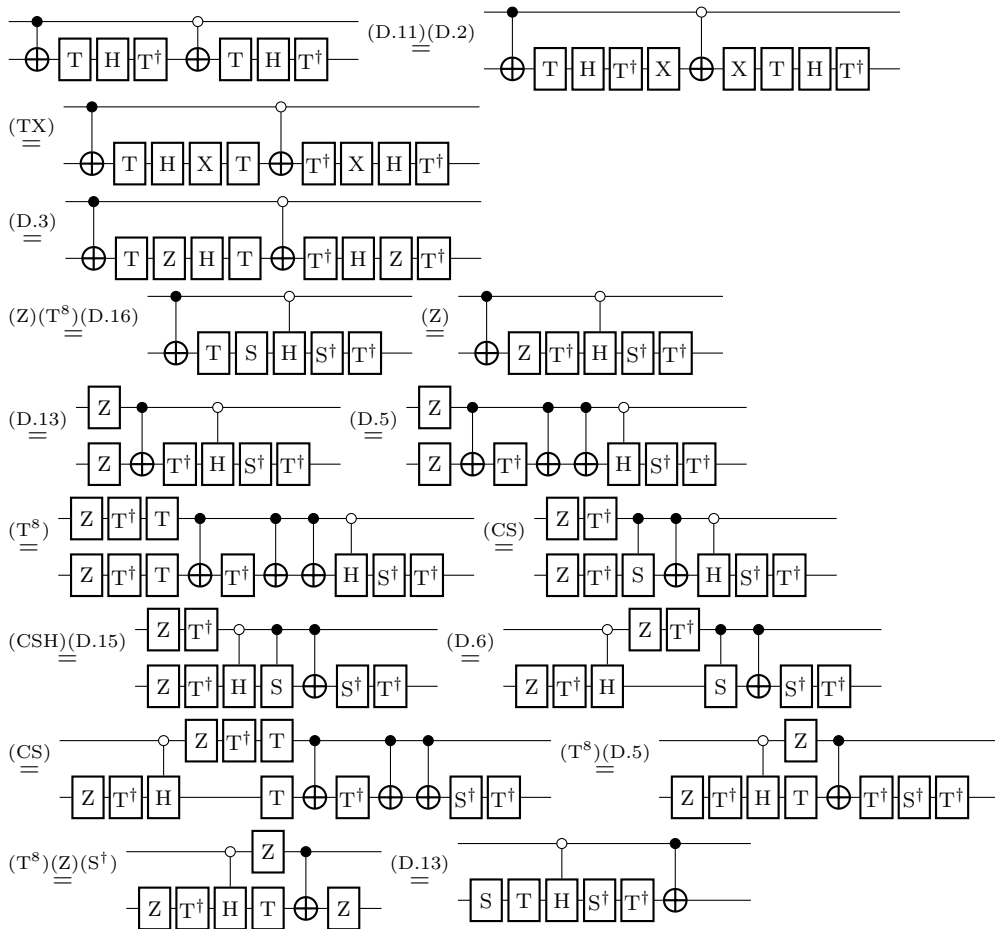




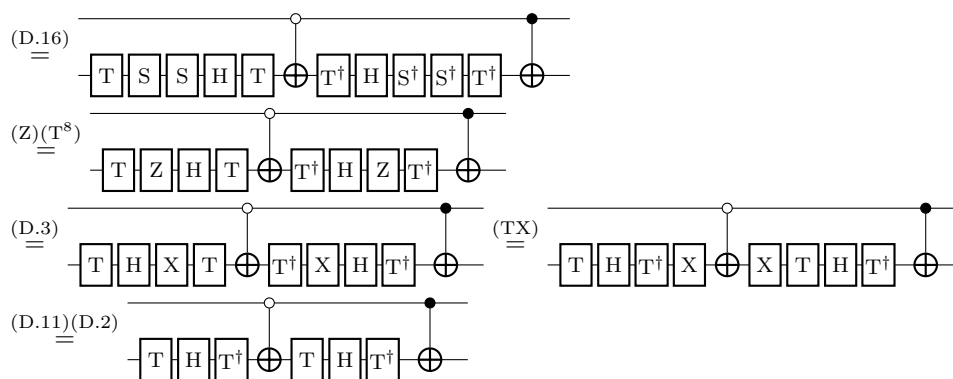
Derivation D.30.



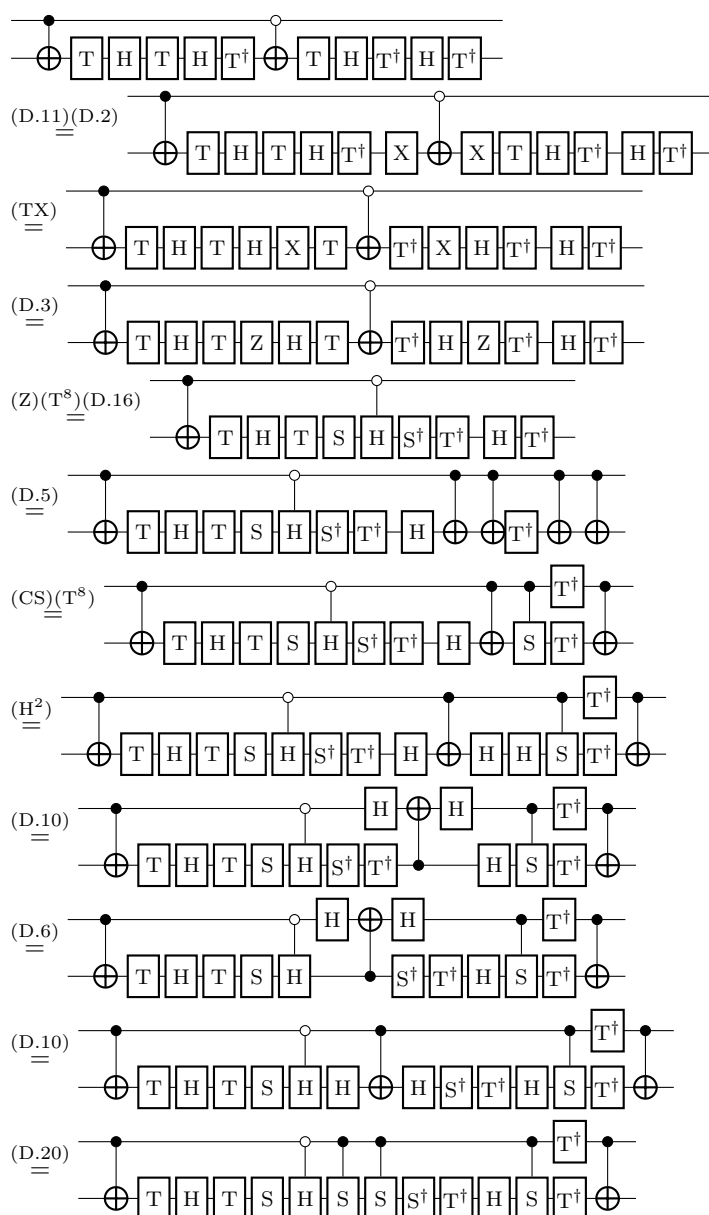
Derivation D.31.

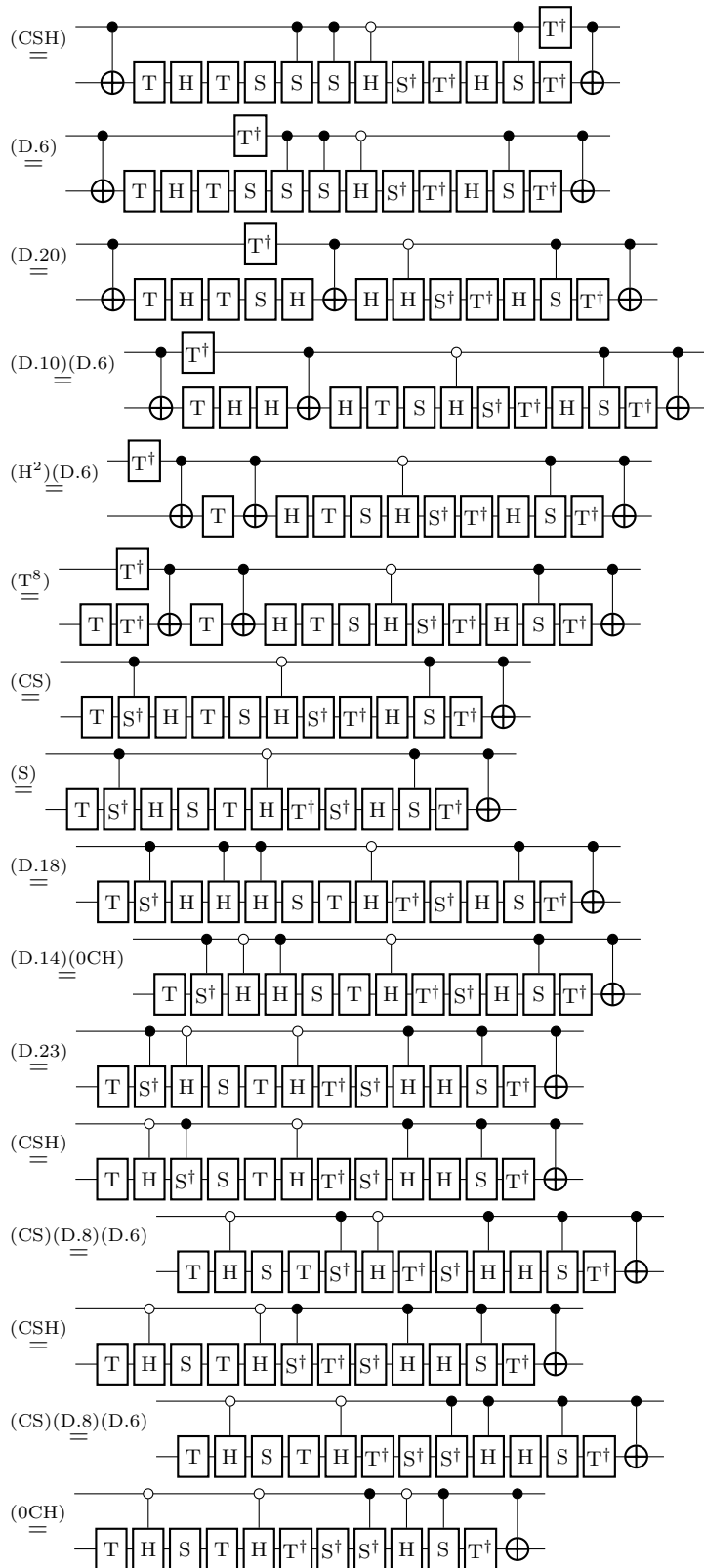


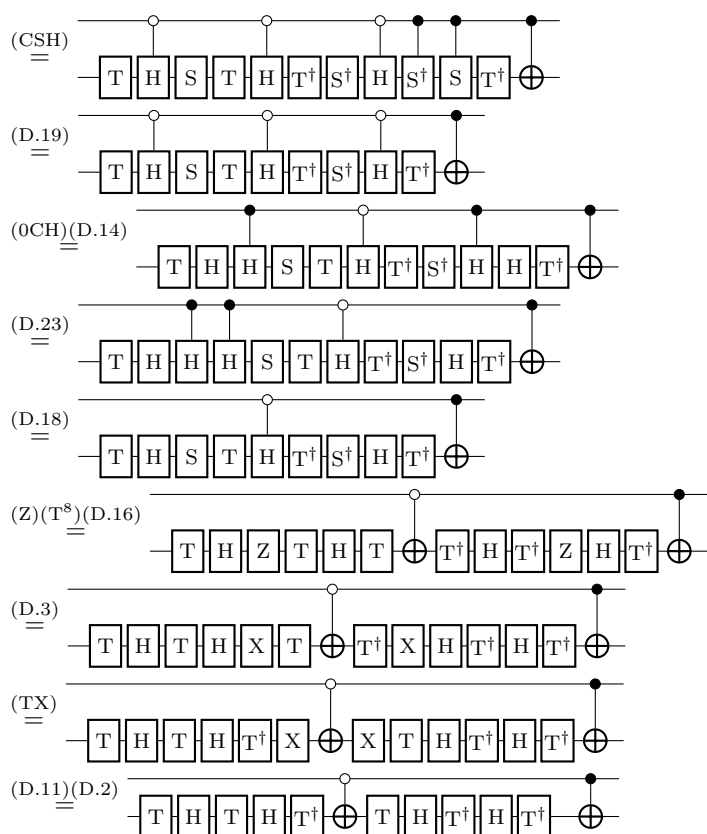
3:44 **Simpler Presentations for Many Fragments of Quantum Circuits**



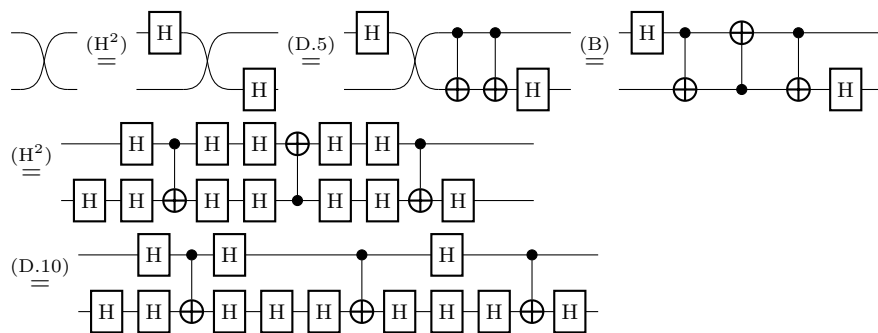
Derivation D.32.





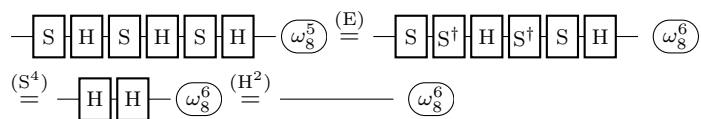


Derivation D.33.

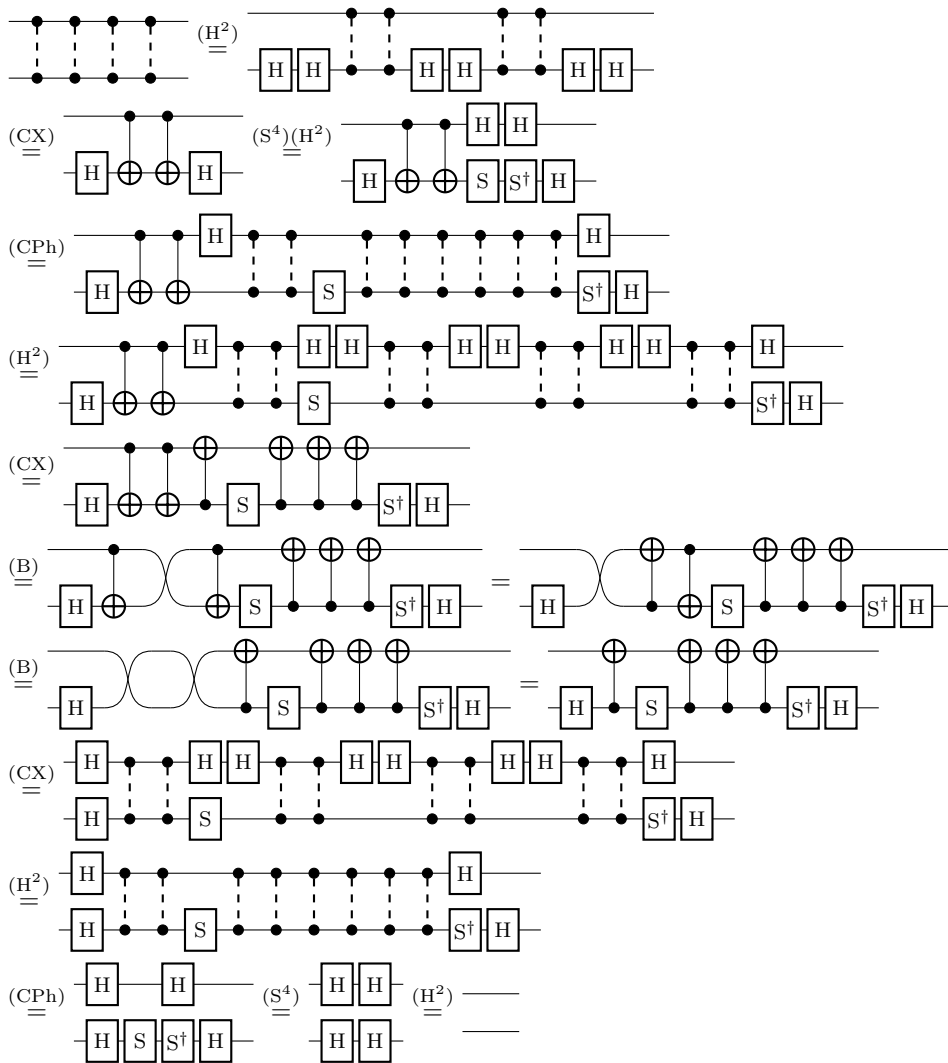


E Auxiliary derivations for CliffCS₂

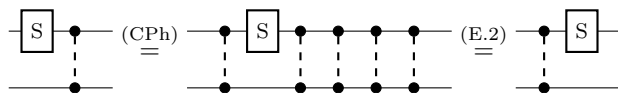
Derivation E.1.



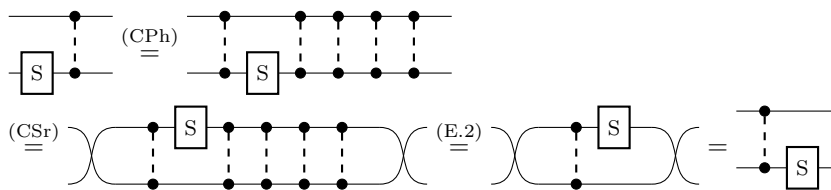
Derivation E.2.



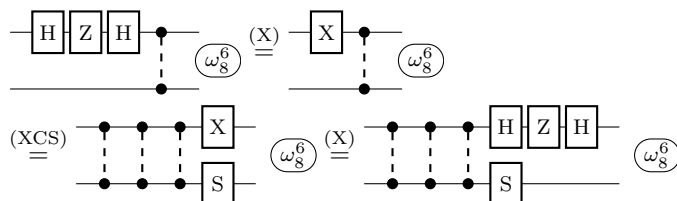
Derivation E.3.



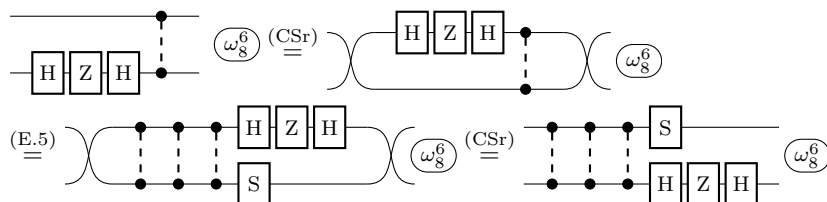
Derivation E.4.



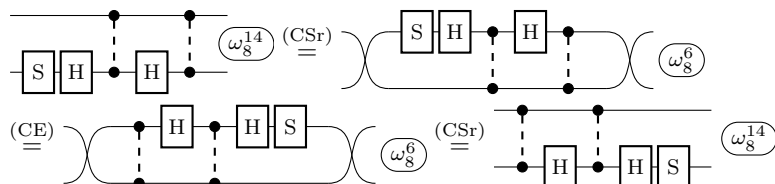
Derivation E.5.



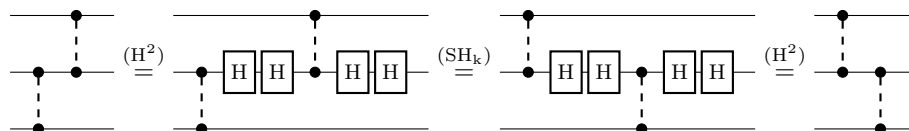
Derivation E.6.



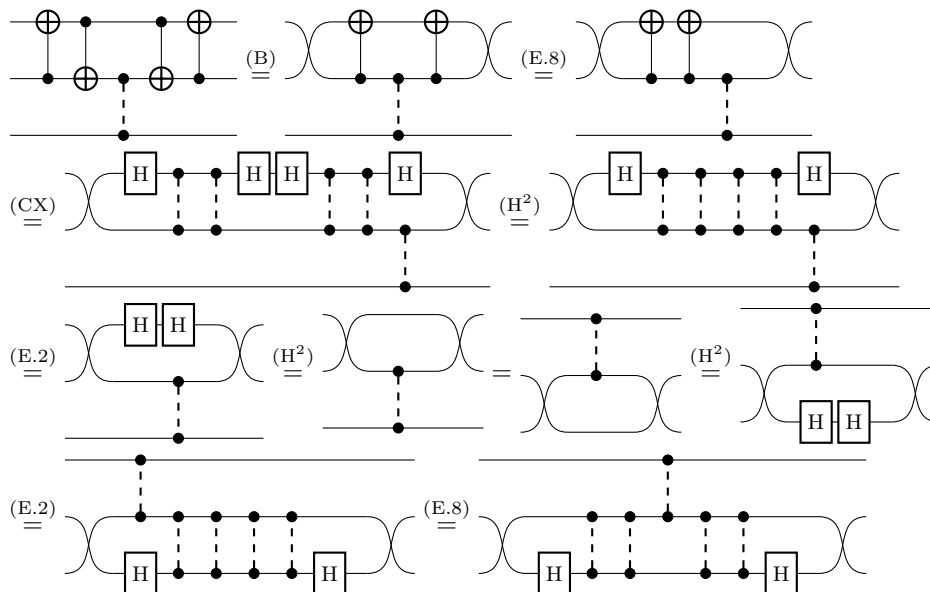
Derivation E.7.

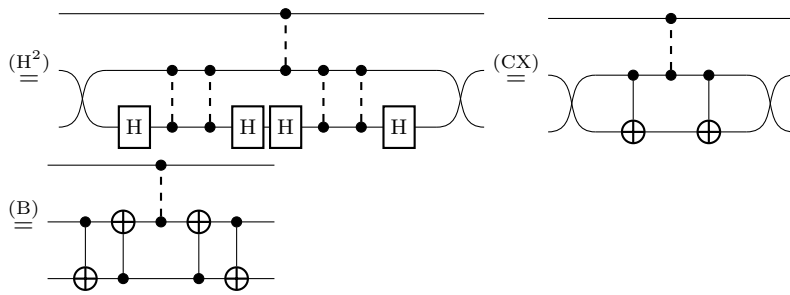


Derivation E.8.

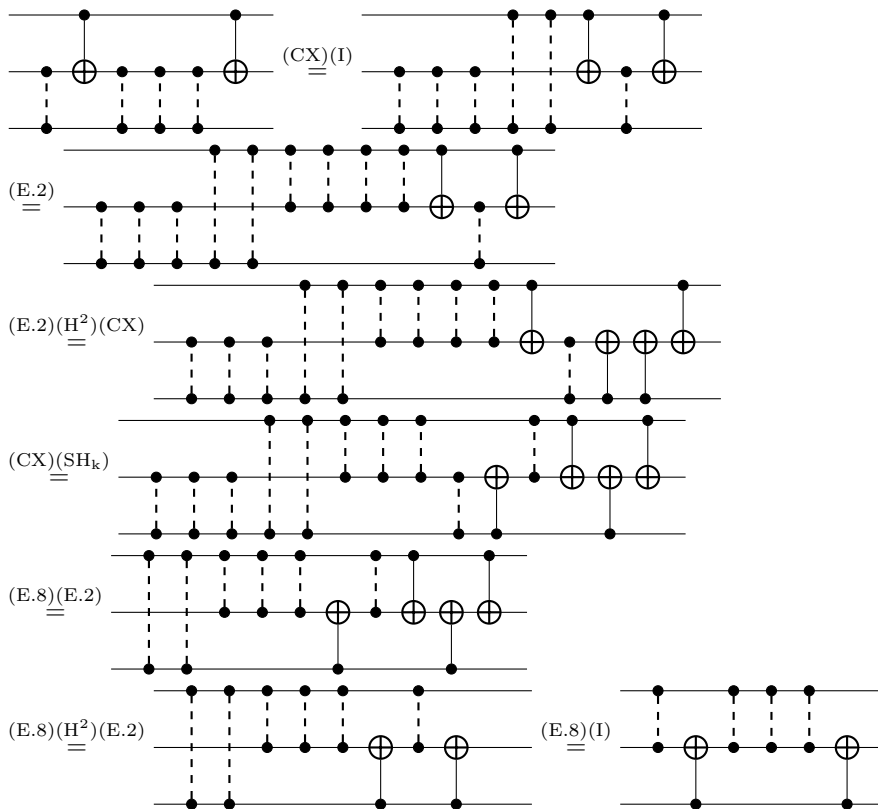


Derivation E.9.

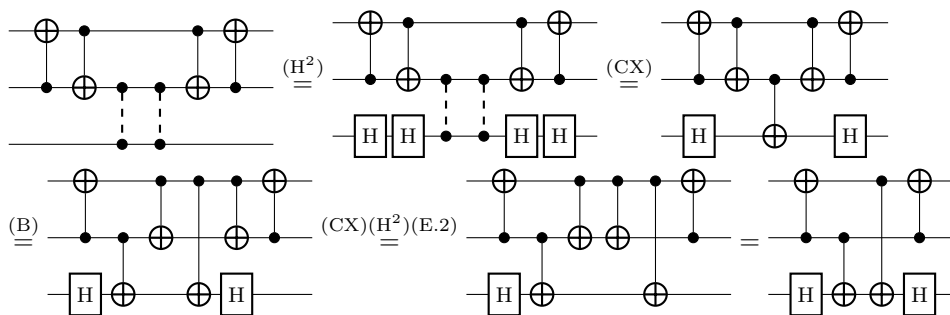




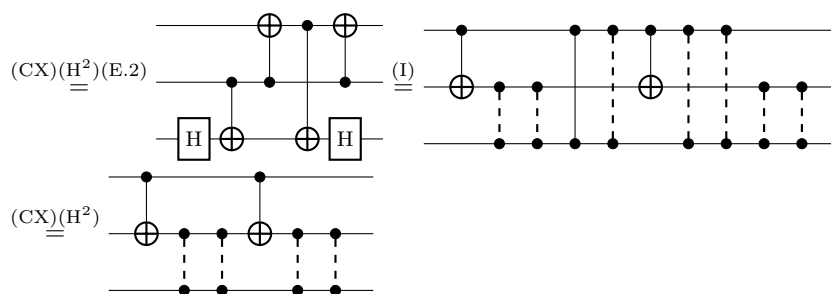
Derivation E.10.



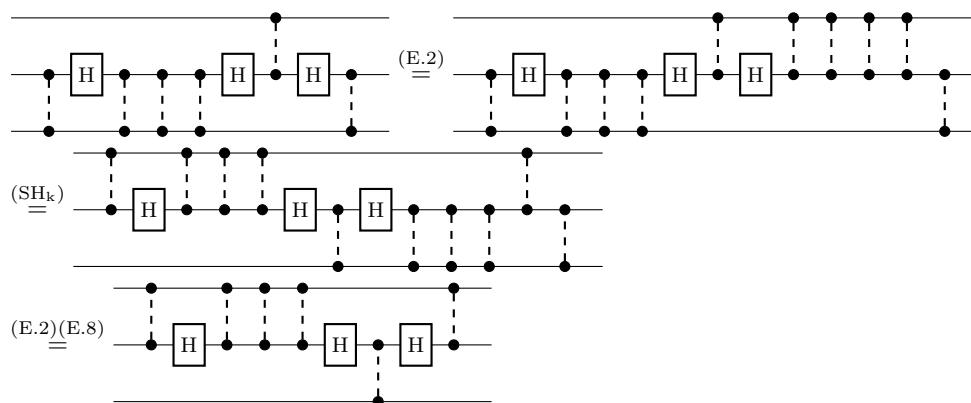
Derivation E.11.



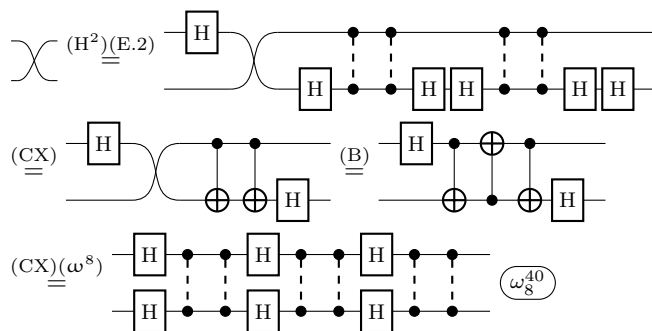
3:50 **Simpler Presentations for Many Fragments of Quantum Circuits**



Derivation E.12.



Derivation E.13.



F Proofs for Lemma 19

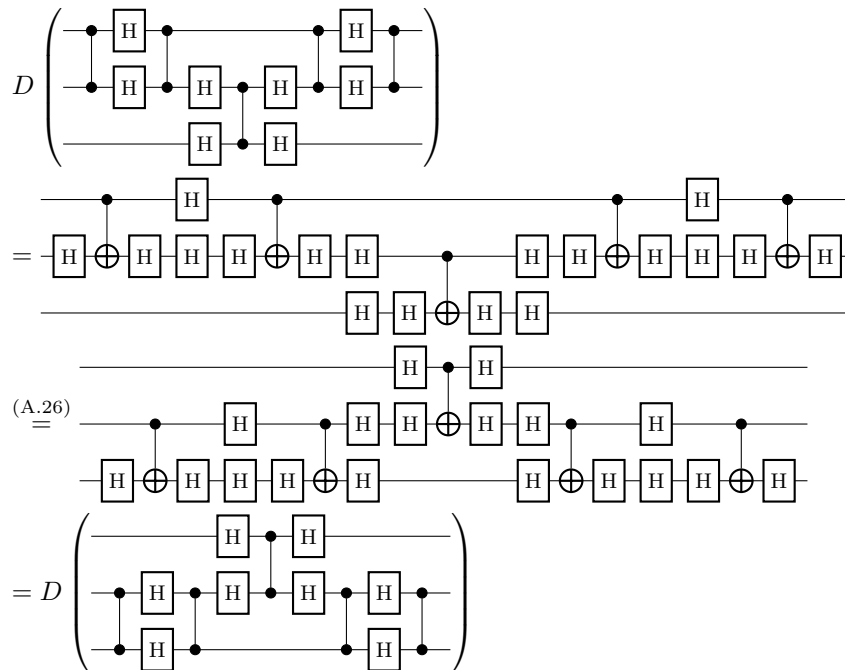
F.1 Clifford

$$D(E(\text{CNOT})) = D(\text{H} \text{---} \text{CNOT} \text{---} \text{H}) = \text{H} \text{---} \text{H} \text{---} \text{CNOT} \text{---} \text{H} \text{---} \text{H} \stackrel{(H^2)}{=} \text{CNOT}.$$

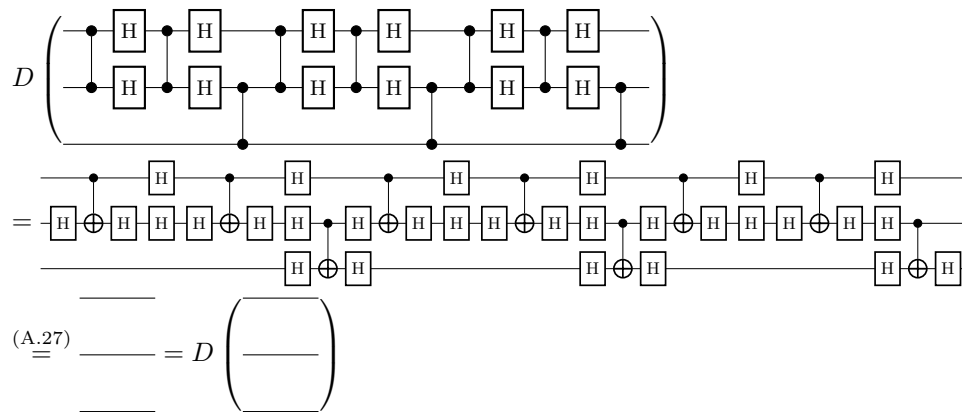
F.2 Real Clifford

$$D(E(\text{CNOT})) = D(\text{H} \text{---} \text{CNOT} \text{---} \text{H}) = \text{H} \text{---} \text{H} \text{---} \text{CNOT} \text{---} \text{H} \text{---} \text{H} \stackrel{(H^2)}{=} \text{CNOT}.$$

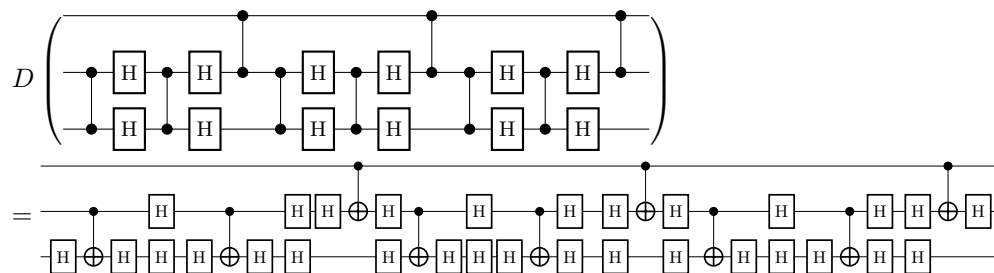
Proof of the decoding of Equation (C13).



Proof of the decoding of Equation (C14).



Proof of the decoding of Equation (C15).



$$\stackrel{(A.28)}{=} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = D \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)$$

G.2 Real-Clifford decoding

Proof of the decoding of Equation (R1), Equation (R2), Equation (R3). Since the decoding only affects the $\begin{array}{c} \text{---} \\ \text{---} \end{array}$ gate, these are trivial.

Proof of the decoding of Equation (R4).

$$D \left(\begin{array}{c} \boxed{Z} \boxed{H} \boxed{Z} \boxed{H} \boxed{Z} \boxed{H} \boxed{Z} \boxed{H} \\ \text{---} \end{array} \right) = \begin{array}{c} \boxed{Z} \boxed{H} \boxed{Z} \boxed{H} \boxed{Z} \boxed{H} \boxed{Z} \boxed{H} \\ \text{---} \end{array} \stackrel{(B.2)}{=} \ominus \text{---} = D \left(\ominus \text{---} \right)$$

Proof of the decoding of Equation (R5).

$$D \left(\begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \bullet \bullet \\ \text{---} \oplus \text{---} \oplus \text{---} \\ \text{---} \end{array} \stackrel{(B.3)}{=} \text{---} = D \left(\text{---} \right)$$

Proof of the decoding of Equation (R6).

$$D \left(\begin{array}{c} \boxed{Z} \bullet \\ \text{---} \bullet \end{array} \right) = \begin{array}{c} \boxed{Z} \bullet \\ \text{---} \oplus \text{---} \oplus \text{---} \end{array} \stackrel{(B.4)}{=} \begin{array}{c} \text{---} \bullet \boxed{Z} \\ \text{---} \oplus \text{---} \oplus \text{---} \end{array} = D \left(\begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \text{---} \end{array} \right)$$

Proof of the decoding of Equation (R7).

$$D \left(\begin{array}{c} \text{---} \bullet \\ \boxed{Z} \bullet \end{array} \right) = \begin{array}{c} \text{---} \bullet \\ \text{---} \oplus \text{---} \oplus \text{---} \end{array} \stackrel{(B.5)}{=} \begin{array}{c} \text{---} \bullet \\ \text{---} \oplus \text{---} \oplus \text{---} \boxed{Z} \end{array} = D \left(\begin{array}{c} \bullet \bullet \\ \bullet \bullet \\ \text{---} \end{array} \right)$$

Proof of the decoding of Equation (R8).

$$D \left(\begin{array}{c} \boxed{H} \boxed{Z} \boxed{H} \bullet \\ \text{---} \bullet \end{array} \right) = \begin{array}{c} \boxed{H} \boxed{Z} \boxed{H} \bullet \\ \text{---} \oplus \text{---} \oplus \text{---} \end{array} \stackrel{(B.6)}{=} \begin{array}{c} \text{---} \bullet \boxed{H} \boxed{Z} \boxed{H} \\ \text{---} \oplus \text{---} \oplus \text{---} \end{array} = D \left(\begin{array}{c} \bullet \bullet \boxed{H} \boxed{Z} \boxed{H} \\ \bullet \bullet \boxed{Z} \end{array} \right)$$

Proof of the decoding of Equation (R9).

$$\begin{aligned}
 D \left(\begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \text{H} \text{---} \text{Z} \text{---} \text{H} \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array} \right) &= \text{---} \text{H} \text{---} \text{Z} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \\
 \stackrel{(B.7)}{=} \begin{array}{c} \text{---} \bullet \text{---} \text{Z} \text{---} \\ | \\ \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{Z} \text{---} \text{H} \text{---} \\ | \\ \text{---} \end{array} &= D \left(\begin{array}{c} \text{---} \bullet \text{---} \text{Z} \text{---} \\ | \\ \text{---} \bullet \text{---} \text{H} \text{---} \text{Z} \text{---} \text{H} \text{---} \\ | \\ \text{---} \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (R10).

$$\begin{aligned}
 D \left(\begin{array}{c} \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array} \right) &= \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\
 &\quad \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \\
 \stackrel{(B.8)}{=} \begin{array}{c} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \\ | \quad | \\ \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{Z} \text{---} \\ | \\ \text{---} \end{array} &= D \left(\begin{array}{c} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \text{H} \text{---} \text{Z} \text{---} \\ | \\ \text{---} \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (R11).

$$\begin{aligned}
 D \left(\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array} \right) &= \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \\
 \stackrel{(B.9)}{=} \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{Z} \text{---} \\ | \quad | \\ \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \\ | \\ \text{---} \end{array} &= D \left(\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{Z} \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \\ | \\ \text{---} \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (R12).

$$\begin{aligned}
 D \left(\begin{array}{c} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ | \quad | \quad | \\ \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ | \\ \text{---} \end{array} \right) &= \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \\
 &\quad \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \\
 \stackrel{(B.10)}{=} \begin{array}{c} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \\ | \quad | \quad | \\ \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \\ | \\ \text{---} \end{array} &= D \left(\begin{array}{c} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \\ | \quad | \quad | \\ \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \\ | \\ \text{---} \end{array} \right)
 \end{aligned}$$



Proof of Equations (S) (R13), (R14), (R15) and (R16). Equations (S), (R13), (R14), (R15) and (R16) of $\text{RCliff}_2^{\text{source}}$ respectively correspond to Equations (S), (C12), (C13), (C14) and (C15) of $\text{Cliff}_2^{\text{source}}$. One could straightforwardly mimic the proofs of Equations C_i to prove Equations R_j .



G.3 Qutrit-Clifford decoding

Proof of the decoding of Equation (C1).

$$D \left(\overline{(-\omega)^6} \right) = \overline{\omega_{12}^{60}} \stackrel{(\omega^{12})}{=} \boxed{\quad} = D \left(\boxed{\quad} \right)$$



Proof of the decoding of Equation (C13).

$$\begin{aligned}
 D \left(\begin{array}{c} \boxed{S} \oplus \\ \bullet \end{array} \right) &= \begin{array}{c} \boxed{S^2} \boxed{H^4} \oplus \boxed{H^4} \\ \bullet \end{array} \quad (\omega_{12}^{40}) \\
 \stackrel{(H^4)(C.25)(\omega^{12})}{=} \begin{array}{c} \boxed{H^4} \oplus \boxed{H^4} \bullet \boxed{S^2} \\ \bullet \boxed{H^3} \oplus \boxed{H} \boxed{S'^2} \end{array} \quad (\omega_{12}^{160}) = D \left(\begin{array}{c} \oplus \bullet \boxed{S} \\ \bullet \bullet \boxed{S'} \end{array} \quad ((-\omega)^8) \right)
 \end{aligned}$$



Proof of the decoding of Equation (C14).

$$\begin{aligned}
 D \left(\begin{array}{c} \bullet \oplus \\ \bullet \bullet \end{array} \right) &= \begin{array}{c} \bullet \boxed{H^4} \oplus \boxed{H^4} \\ \boxed{H^\dagger} \oplus \boxed{H} \bullet \end{array} \quad (\omega_{12}^{36}) \\
 \stackrel{(H^4)(C.26)(\omega^{12})}{=} \begin{array}{c} \oplus \bullet \\ \bullet \boxed{H^3} \oplus \boxed{H} \boxed{S'^2} \boxed{S^2} \end{array} \quad (\omega_{12}^{120}) = D \left(\begin{array}{c} \oplus \bullet \\ \bullet \bullet \boxed{S} \boxed{S'} \end{array} \quad ((-\omega)^4) \right)
 \end{aligned}$$



Proof of the decoding of Equation (C15).

$$D \left(\begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array} \right) = \begin{array}{c} \bullet \oplus \boxed{H^\dagger} \\ \bullet \oplus \boxed{H} \end{array} \stackrel{(C.29)}{=} \begin{array}{c} \bullet \oplus \boxed{H} \\ \bullet \oplus \boxed{H^\dagger} \end{array} = D \left(\begin{array}{c} \bullet \bullet \\ \bullet \bullet \end{array} \right)$$



Proof of the decoding of Equation (C18).

$$\begin{aligned}
 D \left(\begin{array}{c} \bullet \bullet \\ \oplus \bullet \\ \bullet \bullet \end{array} \right) &= \begin{array}{c} \bullet \oplus \boxed{H^4} \\ \bullet \oplus \boxed{H^\dagger} \end{array} \quad (\omega_{12}^{36}) \\
 \stackrel{(I)(H^4)}{=} \begin{array}{c} \bullet \oplus \boxed{H^4} \oplus \boxed{H^4} \\ \bullet \oplus \boxed{H^\dagger} \oplus \boxed{H} \end{array} \quad (\omega_{12}^{36}) = D \left(\begin{array}{c} \bullet \bullet \\ \oplus \bullet \\ \bullet \bullet \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (S).

$$D \left(\begin{array}{c} \text{X} \\ \ominus \end{array} \right) = \begin{array}{c} \boxed{H} \bullet \boxed{H} \bullet \boxed{H} \\ \boxed{H^4} \oplus \boxed{H^5} \oplus \boxed{H^5} \oplus \boxed{H} \end{array} \stackrel{(C.27)(H^4)}{=} \text{X} = D \left(\text{X} \right)$$



Proof of Equations (C11), (C16), (C17), (C7), (C10), (C12). Equation (C11), Equation (C16), Equation (C17), Equation (C7), Equation (C10), Equation (C12) comes from the naturality of the swap and the new rule Equation (S)



G.4 Clifford+T decoding

Proof of the decoding of Equation (C3), Equation (C4), Equation (C15). Since the decoding only affects the \boxed{S} gate and $\text{---}\oplus\text{---}$ gate, these are trivial. ◀

Proof of the decoding of Equation (C5).

$$D(\boxed{S}\boxed{S}\boxed{S}\boxed{S}) = \boxed{T}\boxed{T}\boxed{T}\boxed{T}\boxed{T}\boxed{T}\boxed{T}\boxed{T} \stackrel{(T^8)}{=} \text{---} = D(\text{---})$$

Proof of the decoding of Equation (C6).

$$D(\boxed{S}\boxed{H}\boxed{S}\boxed{H}\boxed{S}\boxed{H}) = \boxed{T^2}\boxed{H}\boxed{T^2}\boxed{H}\boxed{T^2}\boxed{H} \stackrel{(D.24)}{=} \omega_8 \text{---} = D(\omega_8 \text{---})$$

Proof of the decoding of Equation (C14).

$$D(\boxed{S}) = \boxed{T^2} = D(\boxed{T^2})$$

Proof of the decoding of Equation (C7).

$$D(\text{---}\oplus\text{---}) = \boxed{H}\oplus\boxed{H^2}\oplus\boxed{H} \stackrel{(D.25)}{=} \text{---} = D(\text{---})$$

Proof of the decoding of Equation (C8).

$$D(\boxed{S}\text{---}\oplus\text{---}) = \boxed{S}\text{---}\oplus\boxed{H}\oplus\boxed{H} \stackrel{(D.6)}{=} \text{---}\oplus\boxed{H}\oplus\boxed{H}\boxed{S} = D(\text{---}\oplus\boxed{S})$$

Proof of the decoding of Equation (C9).

$$D(\text{---}\oplus\boxed{S}) = \boxed{S}\boxed{H}\oplus\boxed{H} \stackrel{(D.6)(D.10)}{=} \boxed{S}\oplus\boxed{H}\oplus\boxed{H} = D(\text{---}\oplus\boxed{S})$$

Proof of the decoding of Equation (C10).

$$D(\boxed{H}\boxed{S}\boxed{S}\boxed{H}\text{---}\oplus\text{---}) = \boxed{H}\boxed{S}\boxed{S}\boxed{H}\text{---}\oplus\boxed{H}\oplus\boxed{H} \stackrel{(D.26)}{=} \text{---}\oplus\boxed{H}\oplus\boxed{H}\boxed{S}\boxed{S}\boxed{H} = D(\text{---}\oplus\boxed{S}\boxed{S}\boxed{S}\boxed{H})$$

Proof of the decoding of Equation (C18).

$$\begin{aligned}
 D \left(\begin{array}{c} \bullet \\ \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \\ \circ \end{array} \right) &= \begin{array}{c} \bullet \\ \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \\ \circ \end{array} \\
 \stackrel{(D.31)}{=} & \begin{array}{c} \bullet \\ \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \\ \circ \end{array} = D \left(\begin{array}{c} \bullet \\ \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \\ \circ \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (C19).

$$\begin{aligned}
 D \left(\begin{array}{c} \bullet \\ \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \\ \circ \end{array} \right) &= \begin{array}{c} \bullet \\ \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \\ \circ \end{array} \\
 \stackrel{(D.32)}{=} & \begin{array}{c} \bullet \\ \text{---} \text{T} \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \\ \circ \end{array} \\
 &= D \left(\begin{array}{c} \bullet \\ \text{---} \text{T} \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \text{T} \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \text{H} \text{---} \text{T}^\dagger \text{---} \oplus \text{---} \\ \circ \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (C20).

$$\begin{aligned}
 D \left(\begin{array}{c} \circ \\ \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \bullet \\ \oplus \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \circ \end{array} \right) &= \begin{array}{c} \circ \\ \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \bullet \\ \oplus \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \circ \end{array} \\
 \stackrel{(HTH)}{=} & \begin{array}{c} \oplus \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \circ \\ \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \bullet \\ \circ \end{array} = D \left(\begin{array}{c} \oplus \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \circ \\ \text{---} \text{H} \text{---} \text{T} \text{---} \text{H} \text{---} \bullet \\ \circ \end{array} \right)
 \end{aligned}$$



Proof of the decoding of Equation (S).

$$\begin{aligned}
 D(\infty) &= \infty \\
 \stackrel{(D.33)}{=} & \begin{array}{c} \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \text{H} \text{---} \oplus \text{---} \text{H} \text{---} \end{array} = D \left(\begin{array}{c} \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ \oplus \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \end{array} \right)
 \end{aligned}$$



G.5 Clifford+CS decoding

Proof of the decoding of Equation (S).

$$\begin{aligned}
 D(\infty) &= \infty \\
 \stackrel{(E.13)}{=} & \begin{array}{c} \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \\ \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \text{H} \text{---} \bullet \text{---} \end{array} \stackrel{\omega_s^{40}}{=} D \left(\begin{array}{c} \text{---} \text{K} \text{---} \bullet \text{---} \text{K} \text{---} \bullet \text{---} \text{K} \text{---} \bullet \text{---} \\ \text{---} \text{K} \text{---} \bullet \text{---} \text{K} \text{---} \bullet \text{---} \text{K} \text{---} \bullet \text{---} \end{array} \circledast i \right)
 \end{aligned}$$



Proof of the decoding of Equation (C3). Since the decoding does not affect the \square - and \vdash gates, this is trivial. ◀

Proof of the decoding of Equation (C1).

$$D(\textcircled{i^4}) = \textcircled{\omega_8^8} \stackrel{(\omega^8)}{=} \textcircled{\square} = D(\textcircled{\square})$$

Proof of the decoding of Equation (C2).

$$D(\text{---}\square\text{---}) = \text{---}\square\square\text{---} \textcircled{\omega_8^{14}} \stackrel{(H^2)(\omega^8)}{=} \text{---}\textcircled{\omega_8^6}\text{---} = D(\text{---}\textcircled{i^3}\text{---})$$

Proof of the decoding of Equation (C4).

$$D(\text{---}\square\square\square\square\square\square\text{---}) = \text{---}\square\square\square\square\square\square\text{---} \textcircled{\omega_8^{21}} \stackrel{(E.1)(\omega^8)}{=} \text{---}\textcircled{\omega_8^6}\text{---} = D(\text{---}\textcircled{i^3}\text{---})$$

Proof of the decoding of Equation (C5).

$$D(\text{---}\textcircled{\square}\text{---}) = \text{---}\textcircled{\square}\text{---} \stackrel{(E.2)}{=} \text{---}\text{---} = D(\text{---}\text{---})$$

Proof of the decoding of Equation (C6).

$$D(\text{---}\square\text{---}) = \text{---}\square\text{---} \stackrel{(E.3)}{=} \text{---}\text{---}\square\text{---} = D(\text{---}\text{---}\square\text{---})$$

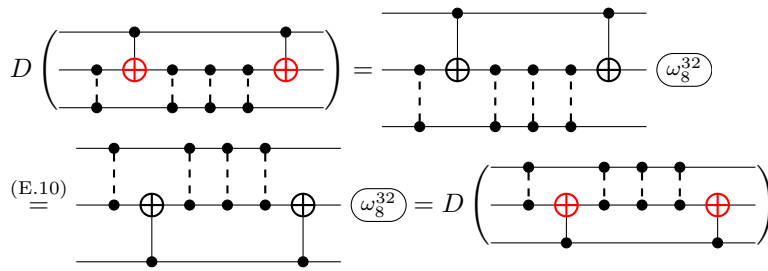
Proof of the decoding of Equation (C7).

$$D(\text{---}\text{---}\square\text{---}) = \text{---}\text{---}\square\text{---} \stackrel{(E.4)}{=} \text{---}\text{---}\text{---}\square\text{---} = D(\text{---}\text{---}\text{---}\square\text{---})$$

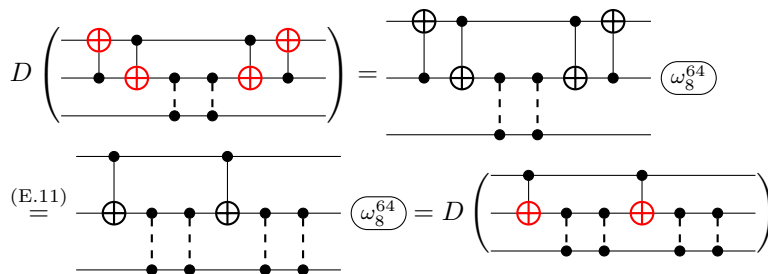
Proof of the decoding of Equation (C8).

$$D(\text{---}\square\square\square\text{---}) = \text{---}\square\square\square\text{---} \textcircled{\omega_8^{14}} \stackrel{(E.5)}{=} \text{---}\square\square\square\text{---} \textcircled{\omega_8^{14}} = D(\text{---}\text{---}\text{---}\square\square\square\text{---})$$

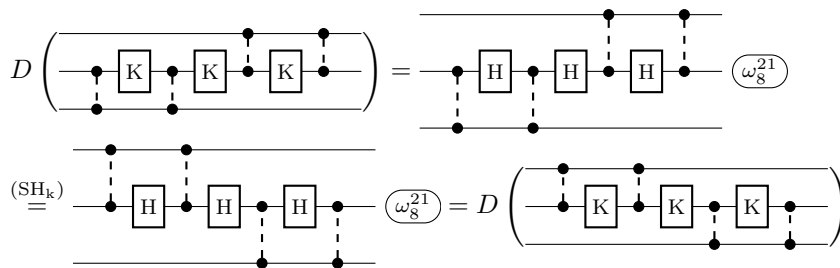
Proof of the decoding of Equation (C14).



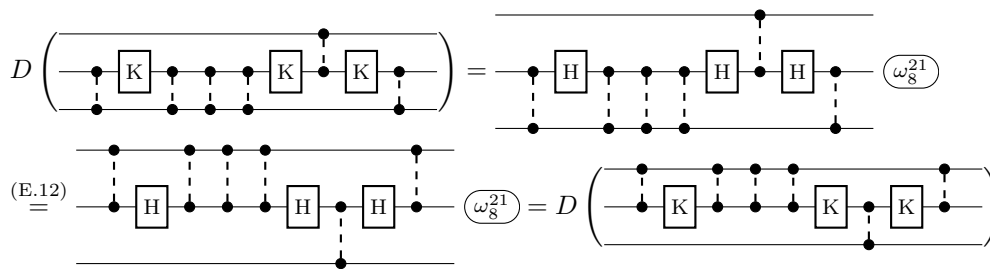
Proof of the decoding of Equation (C15).



Proof of the decoding of Equation (C16).



Proof of the decoding of Equation (C17).



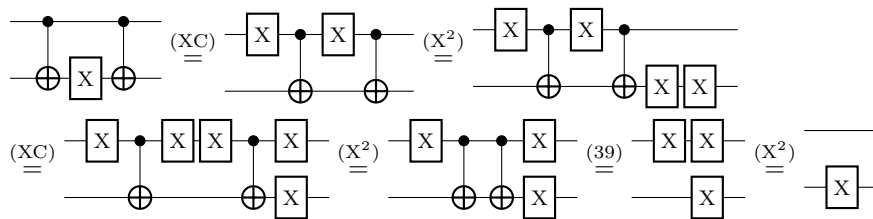
H CNOT-dihedral circuits

For CNOT-dihedral completeness, the fragment-specific work is narrower than in the other cases. No encoding/decoding transfer is needed: the source presentation of 16 already lives in a PROP setting with the same scalar convention as our master signature.

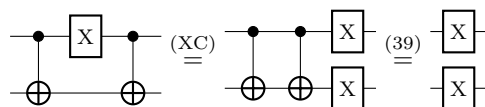
The remaining check is conservative coverage of the source presentation. Every axiom of 16 omitted from the smaller rule set CNOTdihe_2 must be derivable from CNOTdihe_2 itself. The lemmas in this section provide those derivations, so the imported completeness theorem applies unchanged.

► Remark 39. Equation (R4) follows by the same rewriting used in the section for Clifford+T.

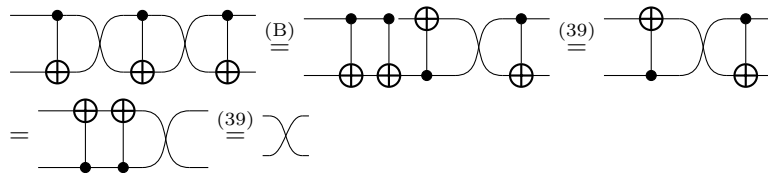
Derivation H.1. Equation (R2)



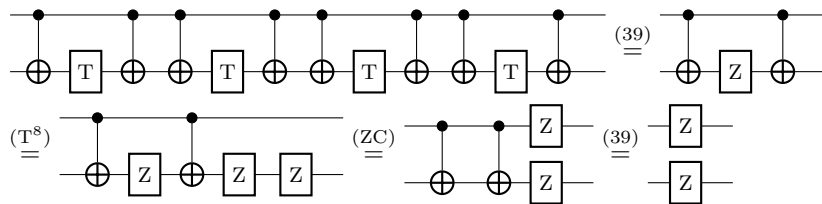
Derivation H.2. Equation (R3)



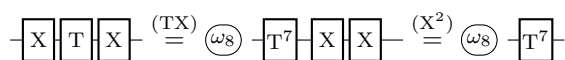
Derivation H.3. Equation (R5)



Derivation H.4. Equation (R8)



Derivation H.5. Equation (R11)



I Scalar refinement

The scalar-refinement step in Section 4.1 addresses a strict-semantics issue: global phases are part of the data of the fragment. Thus two syntactic presentations of the *same* gate fragment

can denote different semantic subPROPs when they generate different subgroups of global phases. Before importing a completeness theorem stated with another scalar convention, we align the scalar subgroups by conservatively extending the syntax with an additional scalar generator.

The main technical claim proved here is that, under mild assumptions satisfied by all our unitary fragments, adjoining a new root-of-unity scalar and relating it to an existing one is a *conservative extension*: it does not introduce any new equalities between circuits that do not mention the new scalar. Equivalently, the induced interpretation remains faithful.

► **Definition 40.** *Let \mathbf{P} be a PROP. Its scalars are the endomorphisms of the tensor unit, $S(\mathbf{P}) := \mathbf{P}(0,0)$. Since 0 is the monoidal unit, both \circ and \otimes restrict to binary operations $S(\mathbf{P}) \times S(\mathbf{P}) \rightarrow S(\mathbf{P})$, and they coincide by interchange. In particular, $S(\mathbf{P})$ is a commutative monoid (and a group whenever \mathbf{P} is a groupoid).*

► **Definition 41.** *Let \mathbf{C} be a (strict) symmetric monoidal subcategory of \mathbf{FdHilb} and assume \mathbf{C} is endomorphism-only: $\mathbf{C}(n,m) = \emptyset$ for $n \neq m$. For $n \in \mathbb{N}$, a unitary $U \in \mathbf{C}(n,n)$ is a global phase on n wires if it is a scalar multiple of the identity, namely $U = \lambda \text{id}_n$ for some $\lambda \in U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. In words: U acts trivially on all states and only contributes a phase factor λ .*

► **Definition 42.** *Let $\mathbf{C} \subseteq \mathbf{FdHilb}$ be a symmetric monoidal subcategory. The visible scalar subgroup of \mathbf{C} is $S(\mathbf{C}) := \mathbf{C}(0,0) \subseteq U(1)$. Every element $\lambda \in S(\mathbf{C})$ yields a global phase on n wires by tensoring: $\lambda \otimes \text{id}_n = \lambda \text{id}_n \in \mathbf{C}(n,n)$.*

► **Definition 43.** *Let $\mathbf{C} \subseteq \mathbf{FdHilb}$ be an endomorphism-only symmetric monoidal category. A phase $\lambda \in U(1)$ is called a hidden phase of \mathbf{C} if there exists some arity $n \geq 1$ such that $\lambda \text{id}_n \in \mathbf{C}(n,n)$ while $\lambda \notin S(\mathbf{C})$.*

Equivalently, λ is hidden if it occurs as a global phase at some positive wire count, but cannot be produced as a scalar morphism $0 \rightarrow 0$ inside \mathbf{C} .

► **Example 44.** Consider the subgroup $\mathbf{C} \subseteq \mathbf{Qubit}$ generated (as a symmetric monoidal subcategory) by a single 1-qubit gate $R_X(2\pi)$, but with no scalar generators. Semantically, $R_X(2\pi) = -\text{id}_1$, hence $-\text{id}_1 \in \mathbf{C}(1,1)$. However, $S(\mathbf{C}) = \{1\}$ because there are no nontrivial $0 \rightarrow 0$ morphisms available. Thus $\lambda = -1$ is a hidden phase in the sense of Definition 43.

If we were to adjoin a new scalar generator ω interpreted as -1 , then $\omega \otimes \text{id}_1$ and the existing circuit $R_X(2\pi)$ would have the same semantics; without additional axioms linking them, this would typically destroy faithfulness of the interpretation. The “no hidden phases” assumption in Lemma 50 precisely rules out this situation.

The conservativity argument relies on the observation that scalars commute with all endomorphisms, so new scalar generators can always be “pulled out” as a single global phase factor.

► **Lemma 45.** *Let \mathbf{C} be a strict symmetric monoidal category. For any scalar $a : 0 \rightarrow 0$ and any endomorphism $f : n \rightarrow n$ one has*

$$(a \otimes \text{id}_n) \circ f = f \circ (a \otimes \text{id}_n). \quad (5)$$

For scalars $a, b : 0 \rightarrow 0$, composition and tensor coincide: $a \circ b = a \otimes b$.

Proof. Since the tensor is bifunctorial and the category is strict, $(a \otimes \text{id}_n) \circ f = (a \circ \text{id}_0) \otimes (\text{id}_n \circ f) = (\text{id}_0 \circ a) \otimes (f \circ \text{id}_n) = f \circ (a \otimes \text{id}_n)$, using the interchange law. For scalars, the same computation with $n = 0$ yields $a \circ b = a \otimes b$. ◀

The scalar-refinement step is an extension of a presentation by adjoining a new scalar generator of higher order.

► **Definition 46.** For $m \geq 1$ write $\mu_m := \{e^{2\pi ik/m} \mid 0 \leq k < m\} \subseteq U(1)$ for the cyclic group of m -th roots of unity.

► **Definition 47.** Let $\mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}}$ be a presented PROP and let $s : 0 \rightarrow 0$ be a chosen scalar in the src signature. Fix integers $m \geq 1$ and $\ell \geq 1$. Define $\mathbf{P}^{\text{src},\sharp}/\mathcal{R}^{\text{src},\sharp}$ to be the presentation obtained by adjoining a new scalar generator $\omega : 0 \rightarrow 0$ and adding the two relations

$$\omega^{m\ell} = \text{id}_0, \quad \omega^r = s, \quad (6)$$

where ω^k denotes the k -fold composite of ω (equivalently, k -fold tensor), and r is an integer chosen so that semantically ω^r matches s as in (8).

► **Remark 48.** The relations (6) assert that ω has order dividing $m\ell$, and that the src scalar s is the r -th power of ω . Concretely, this is the standard “adjoin a root” construction at the level of scalars. The point of Lemma 50 is that, in a PROP setting, adjoining such a root is conservative provided there are no hidden phases.

The centrality of scalars yields a simple normal form in the refined syntax: all occurrences of ω can be extracted into a single scalar factor.

► **Lemma 49.** In the refined PROP $\mathbf{P}^{\text{src},\sharp}/\mathcal{R}^{\text{src},\sharp}$, every morphism $C : n \rightarrow n$ is derivably equal to one of the form

$$C = \omega^a \otimes C', \quad (7)$$

where $a \in \{0, 1, \dots, m\ell - 1\}$ and C' is a morphism built using only the src generators (i.e. C' contains no occurrence of ω).

Proof. Work in the free PROP on the refined signature, modulo PROP coherence and the relations \mathcal{R}^{src} and (6). Because ω is a scalar, every occurrence of ω in a diagram is a $0 \rightarrow 0$ component tensored somewhere with an endomorphism. By Lemma 45, we can slide each such component past all other boxes and past all compositions/tensors so as to collect all ω -factors together as a single power ω^a tensored with a diagram C' that contains no ω . Finally, use $\omega^{m\ell} = \text{id}_0$ to reduce a modulo $m\ell$ into the stated range. ◀

The main lemma used in Section 4.1 is the resulting conservativity statement.

► **Lemma 50.** Let $\mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}}$ be a graphical language equipped with a faithful (i.e. complete) strict symmetric monoidal interpretation $\llbracket \cdot \rrbracket_{\text{src}} : \mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}} \rightarrow \mathbf{C}^{\text{src}} \subseteq \mathbf{FdHilb}$. Assume the following semantic properties.

1. (finite cyclic visible scalars) The scalar group $S(\mathbf{C}^{\text{src}}) = \mathbf{C}^{\text{src}}(0, 0)$ is finite cyclic of order m and generated by $\llbracket s \rrbracket_{\text{src}}$ for some chosen src scalar $s : 0 \rightarrow 0$. Equivalently, $S(\mathbf{C}^{\text{src}}) \cong \mu_m$.
2. (no hidden phases) \mathbf{C}^{src} has no hidden phases: for all $n \in \mathbb{N}$ and all $\lambda \in U(1)$, if $\lambda \text{id}_n \in \mathbf{C}^{\text{src}}(n, n)$ then $\lambda \in S(\mathbf{C}^{\text{src}})$ (cf. Definition 43).
3. (invertibility) Every morphism in \mathbf{C}^{src} is invertible, i.e. each hom-set $\mathbf{C}^{\text{src}}(n, n)$ is a group under composition.²

² In all applications in this paper, \mathbf{C}^{src} consists of unitaries, hence this holds automatically.

Fix $\ell \geq 1$ and choose a primitive root of unity $\zeta \in \mathbb{U}(1)$ of order $m\ell$. Choose an integer r such that

$$\zeta^r = \llbracket s \rrbracket_{\text{src}}. \quad (8)$$

Form the scalar-refined presentation $\mathbf{P}^{\text{src},\sharp}/\mathcal{R}^{\text{src},\sharp}$ as in Definition 47.

Let $\mathbf{C}^\sharp \subseteq \mathbf{FdHilb}$ be the subPROP generated by \mathbf{C}^{src} together with the scalar $\zeta : 0 \rightarrow 0$ (viewed as the unitary map $\mathbb{C} \rightarrow \mathbb{C}$, $1 \mapsto \zeta$).

Then the interpretation functor extends uniquely to a strict symmetric monoidal functor $\llbracket \cdot \rrbracket^\sharp : \mathbf{P}^{\text{src},\sharp}/\mathcal{R}^{\text{src},\sharp} \rightarrow \mathbf{C}^\sharp$ sending ω to ζ , and $\llbracket \cdot \rrbracket^\sharp$ is faithful.

In particular, the refined language is complete for \mathbf{C}^\sharp and $S(\mathbf{C}^\sharp) \cong \mu_{m\ell}$.

Proof. (1) Soundness and existence of the refined interpretation. By construction, $\llbracket \cdot \rrbracket^\sharp$ agrees with $\llbracket \cdot \rrbracket_{\text{src}}$ on all src generators. We interpret the new generator ω by the scalar ζ . The additional relations (6) hold semantically because $\zeta^{m\ell} = 1$ and $\zeta^r = \llbracket s \rrbracket_{\text{src}}$ by (8). Therefore the universal property of presented PROPs yields a unique induced functor $\llbracket \cdot \rrbracket^\sharp$ into \mathbf{C}^\sharp .

(2) Scalar extraction in the refined syntax. By Lemma 49, every refined circuit $C : n \rightarrow n$ is derivably equal to $\omega^a \otimes C'$ with $0 \leq a < m\ell$ and C' an src circuit.

(3) Faithfulness. Let $C_1, C_2 : n \rightarrow n$ be refined circuits such that $\llbracket C_1 \rrbracket^\sharp = \llbracket C_2 \rrbracket^\sharp$. Write both in scalar-extraction normal form: $C_i = \omega^{a_i} \otimes C'_i$ ($0 \leq a_i < m\ell$), with C'_i containing no ω . Let $U_i := \llbracket C'_i \rrbracket_{\text{src}} \in \mathbf{C}^{\text{src}}(n, n)$. The semantic equality becomes

$$\zeta^{a_1} U_1 = \zeta^{a_2} U_2. \quad (9)$$

By invertibility 3, we can right-multiply by U_2^{-1} to obtain

$$\zeta^{a_2 - a_1} \text{id}_n = U_1 \circ U_2^{-1} \in \mathbf{C}^{\text{src}}(n, n). \quad (10)$$

Thus $\zeta^{a_2 - a_1}$ appears as a global phase in \mathbf{C}^{src} on n wires. By the no-hidden-phases assumption 2, this implies $\zeta^{a_2 - a_1} \in S(\mathbf{C}^{\text{src}})$. By 1, there exists $t \in \{0, \dots, m-1\}$ such that

$$\zeta^{a_2 - a_1} = \llbracket s \rrbracket_{\text{src}}^t. \quad (11)$$

Combining (9) with (11) gives $\llbracket C_1 \rrbracket_{\text{src}} = \llbracket s^t \otimes C'_2 \rrbracket_{\text{src}}$. Since $\llbracket \cdot \rrbracket_{\text{src}}$ is faithful by hypothesis, $\mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}} \vdash C_1 = s^t \otimes C'_2$. The refined presentation contains all src relations, so the same derivation is valid in $\mathbf{P}^{\text{src},\sharp}/\mathcal{R}^{\text{src},\sharp}$. Now use the refinement relation $\omega^r = s$ to rewrite $s^t = \omega^{rt}$, yielding $C_1 = \omega^{a_1} \otimes C'_1 = \omega^{a_1} \otimes \omega^{rt} \otimes C'_2 = \omega^{a_1 + rt} \otimes C'_2$.

It remains to compare the exponents. From (11) and (8) we have $\zeta^{a_2 - a_1} = \zeta^{rt}$, so $\zeta^{a_2 - a_1 - rt} = 1$. Since ζ has exact order $m\ell$, this implies $a_2 - a_1 - rt \equiv 0 \pmod{m\ell}$. Using $\omega^{m\ell} = \text{id}_0$ in the refined syntax, we conclude $\omega^{a_1 + rt} = \omega^{a_2}$, hence $C_1 = \omega^{a_2} \otimes C'_2 = C_2$. Therefore $\llbracket \cdot \rrbracket^\sharp$ is faithful.

(4) Scalars of \mathbf{C}^\sharp . By definition, $S(\mathbf{C}^\sharp)$ is generated by $S(\mathbf{C}^{\text{src}}) = \langle \llbracket s \rrbracket_{\text{src}} \rangle$ together with ζ . Since $\llbracket s \rrbracket_{\text{src}} = \zeta^r \in \langle \zeta \rangle$, we obtain $S(\mathbf{C}^\sharp) = \langle \zeta \rangle \cong \mu_{m\ell}$. ◀

► **Corollary 51.** *With the hypotheses of Lemma 50, let C_1, C_2 be src circuits (i.e. containing no occurrence of ω). If $\mathbf{P}^{\text{src},\sharp}/\mathcal{R}^{\text{src},\sharp} \vdash C_1 = C_2$, then already $\mathbf{P}^{\text{src}}/\mathcal{R}^{\text{src}} \vdash C_1 = C_2$.*

Proof. If the refined theory derives $C_1 = C_2$, then in particular $\llbracket C_1 \rrbracket_{\text{src}} = \llbracket C_2 \rrbracket_{\text{src}}$ (because the refined interpretation restricts to the src one on src syntax). Faithfulness of $\llbracket \cdot \rrbracket_{\text{src}}$ implies the src derivability. ◀

The two scalar refinements applied in the main text are the following.

| Fragment | Encoding E_\bullet (new \rightarrow src) | Decoding D_\bullet (src \rightarrow new) |
|-----------------|---|---|
| $Cliff, RCliff$ | $E_\bullet \left(\begin{array}{c} \bullet \\ \oplus \end{array} \right) = \begin{array}{c} \bullet \\ \text{H} \text{---} \bullet \text{---} \text{H} \end{array}$ | $D_\bullet \left(\begin{array}{c} \bullet \\ \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \text{H} \oplus \text{H} \end{array}$ |
| $CliffT$ | $E_{CliffT} \left(\begin{array}{c} \bullet \\ \oplus \end{array} \right) = \begin{array}{c} \bullet \\ \text{H} \text{---} \bullet \text{---} \text{H} \end{array}$ | $D_{CliffT} \left(\begin{array}{c} \bullet \\ \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \text{H} \oplus \text{H} \end{array}$ $D_{CliffT} \left(\begin{array}{c} \text{S} \end{array} \right) = \begin{array}{c} \text{T}^2 \end{array}$ |
| $Cliff3$ | $E_{Cliff3} \left(\begin{array}{c} \text{S} \end{array} \right) = \begin{array}{c} \text{S}^2 \text{---} \omega_{12}^4 \end{array}$ $E_{Cliff3} \left(\begin{array}{c} \text{H} \end{array} \right) = \begin{array}{c} \text{H} \text{---} \omega_{12}^3 \end{array}$ $E_{Cliff3} \left(\begin{array}{c} \bullet \\ \oplus \end{array} \right) = \begin{array}{c} \bullet \\ \text{H} \text{---} \bullet \text{---} \text{H}^3 \end{array}$ $E_{Cliff3} \left(\omega_{12} \right) = \omega_{12}$ | $D_{Cliff3} \left(\begin{array}{c} \text{S} \end{array} \right) = \begin{array}{c} \text{S}^2 \text{---} \omega_{12}^4 \end{array}$ $D_{Cliff3} \left(\begin{array}{c} \text{H} \end{array} \right) = \begin{array}{c} \text{H} \text{---} \omega_{12}^9 \end{array}$ $D_{Cliff3} \left(\begin{array}{c} \bullet \\ \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \text{H}^3 \oplus \text{H} \end{array}$ $D_{Cliff3} \left(\omega_{12} \right) = \omega_{12}$ $D_{Cliff3} \left(-\omega \right) = \omega_{12}^{10}$ |
| $CliffCS$ | $E_{CliffCS} \left(\begin{array}{c} \text{H} \end{array} \right) = \begin{array}{c} \text{K} \text{---} \omega_8 \end{array}$ $E_{CliffCS} \left(\omega_8 \right) = \omega_8$ | $D_{CliffCS} \left(\begin{array}{c} \text{K} \end{array} \right) = \begin{array}{c} \text{H} \text{---} \omega_8^7 \end{array}$ $D_{CliffCS} \left(i \right) = \omega_8^2$ $D_{CliffCS} \left(\omega_8 \right) = \omega_8$ |

■ **Table 3** Encoding/decoding pairs used in the completeness-transfer arguments. For the qutrit Clifford and Clifford+CS rows, the source presentation is the scalar-refined one from Lemma 50. *Convention:* omitted generators are fixed by E_\bullet and D_\bullet .

- **Qutrit Clifford.** The imported presentation of [?] uses an order-6 scalar $(-\omega)$. Our strict qutrit convention uses an order-12 scalar subgroup (Section 3). We refine the source syntax by adjoining an order-12 scalar generator (ω_{12}) and adding the relations $(\omega_{12})^{12} = \text{id}_0$ and $(\omega_{12})^{10} = (-\omega)$. This is the $\mu_6 \rightarrow \mu_{12}$ refinement referenced in Section 4.1.
- **Clifford+CS.** The imported presentation of [?] uses an order-4 scalar (i) , whereas our strict qubit convention uses an order-8 scalar subgroup. We refine the source syntax by adjoining an order-8 scalar generator (ω_8) and adding the relations $(\omega_8)^8 = \text{id}_0$ and $(\omega_8)^2 = (i)$. This is the $\mu_4 \rightarrow \mu_8$ refinement referenced in Section 4.1.

In both cases, the hypotheses of Lemma 50 are satisfied for the unitary semantics considered in this paper: invertibility is automatic, visible scalars are finite cyclic by construction, and the imported normal forms expose every scalar multiple of an identity as one of those visible scalars, so no hidden phases appear relative to the chosen scalar conventions. Therefore, scalar refinement preserves faithfulness, and the refined imported presentations may be treated as complete presentations for the strict semantics used throughout.

| Interpretation | Cliff | Real Cliff | Qutrit Cliff | Cliff+T | Cliff+CS | CNOT-dih | Arity |
|---|--------------|------------|-----------------|--------------|--------------|--------------|-------|
| $?\otimes$ | (ω^8) | — | — | (ω^8) | (ω^8) | (ω^8) | 0 |
| $?\ominus$ | — | $(-^2)$ | — | — | — | — | 0 |
| $?\omega_{12}$ | — | — | (ω^{12}) | — | — | — | 0 |
| $?\text{-H-}$ | (H^2) | (H^2) | (H^4) | (H^2) | (H^2) | — | 1 |
| $?\text{-S-}$ | (S^4) | — | (S^3) | — | (S^4) | — | 1 |
| $?\text{-Z-}$ | — | (Z^2) | — | — | — | — | 1 |
| $?\text{-T-}$ | — | — | — | (T^8) | — | (T^8) | 1 |
| $?\text{-X-}$ | — | — | — | — | — | (X^2) | 1 |
| $\#\{\text{-H-}\}_{[2]}$ | (E) | — | (E) | (E) | (E) | — | 1 |
| $\#\{\otimes\}_{[2]}$ | — | — | — | — | — | (TX) | 1 |
| $\#\{\ominus\}_{[2]}$ | — | (F) | — | — | — | — | 1 |
| $\#\{\text{-H-}, \otimes\}_{[2]}$ | — | — | — | (TX) | — | — | 1 |
| $\#\{\text{-X-}\}_{[2]}$ | — | — | — | — | — | (XC) | 1 |
| $[\text{-S-} := \text{-S-}\text{-X-}] \sim$ | — | — | $(SS')^a$ | — | — | — | 1 |
| $?\oplus$ | (CPh) | (CX^2) | (CPh) | (CPh) | — | (CPh) | 2 |
| $?\text{-}$ | — | — | — | — | (CPh) | — | 2 |
| $\#\{\text{-}\}_{[2]}$ | (B) | (B) | (B) | (B) | (B) | (B) | 2 |
| $\#\{\oplus, \text{-}\}_{[2]}$ | (CZ) | — | — | (CZ) | — | — | 2 |
| $\#\{\text{-S-}\}_{[3]}$ | — | — | (CZ) | — | — | — | 2 |
| $\#\{\text{-Z-}\}_{[2]}$ | — | (ZC) | — | — | — | — | 2 |
| $[\text{-Z-} := \text{-}] \sim$ | — | (CF) | — | — | — | — | 2 |
| $[\text{-H-} := \text{-}] \sim$ | — | (CZr) | — | — | — | — | 2 |
| $\#\{\oplus\}_{[2]}$ | — | — | (KC) | — | — | — | 2 |
| $\#\{\text{-S-}, \text{-H-}\}_{[2]}$ | — | — | — | — | (XCS) | — | 2 |
| $\text{-} := \text{-Z-} \text{-} \sim$ | — | — | — | — | (CSr) | — | 2 |
| $\text{-} := \text{-Z-} \text{-} \sim$ | — | — | — | — | (CE) | — | 2 |
| $\#\{\text{-T-}, \otimes, \otimes\}_{[8]}$ | — | — | — | — | — | (ZC) | 2 |
| arg det ₂ | (I) | (I) | — | — | (I) | — | 3 |
| $\#\{\oplus, \text{-}\}_{[2]}$ | — | — | — | — | — | (I) | 3 |
| $\#\{\text{-T-}, \otimes, \otimes\}_{[4]}$ | — | — | — | — | — | (C^2T) | 3 |
| arg det ₃ | — | — | — | — | — | (C^3T) | 4 |

a) See Proposition 38.

■ **Figure 11** Summary of minimality interpretations across all fragments

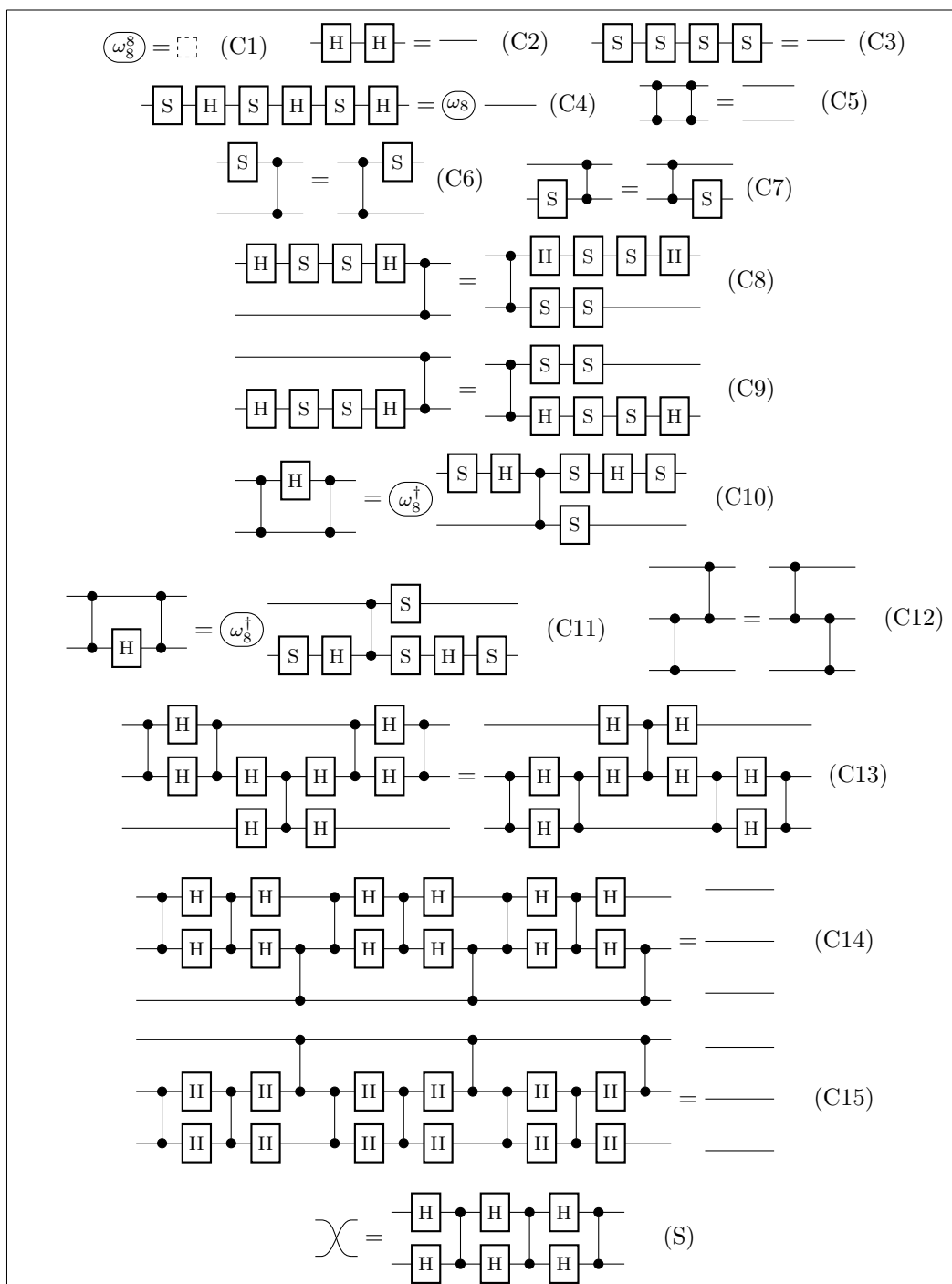


Figure 12 Set of relations $\text{Cliff}_2^{\text{source}}$. Equations (C_i) are the ones of the complete set of relations for the pro considered in [?].

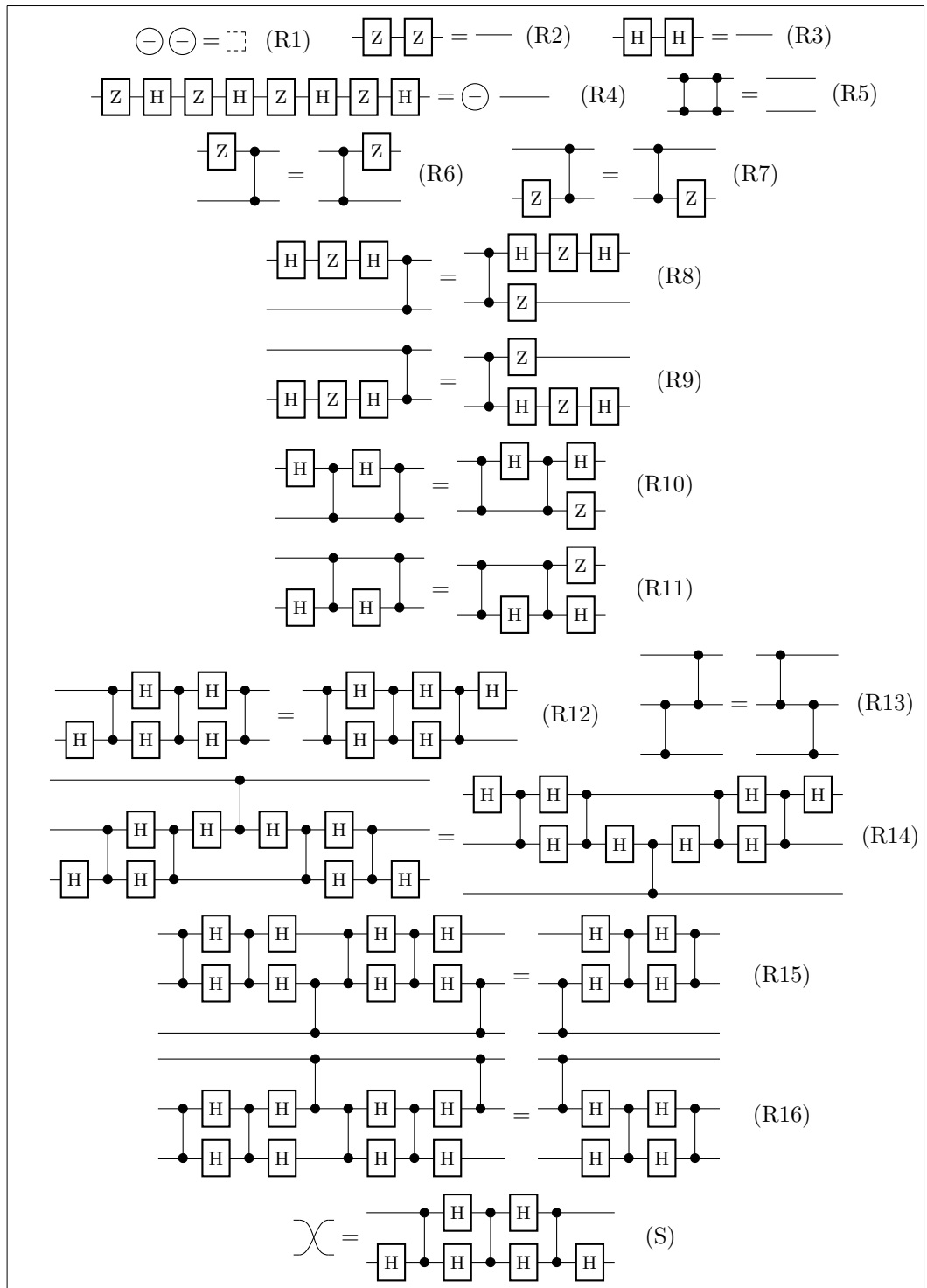
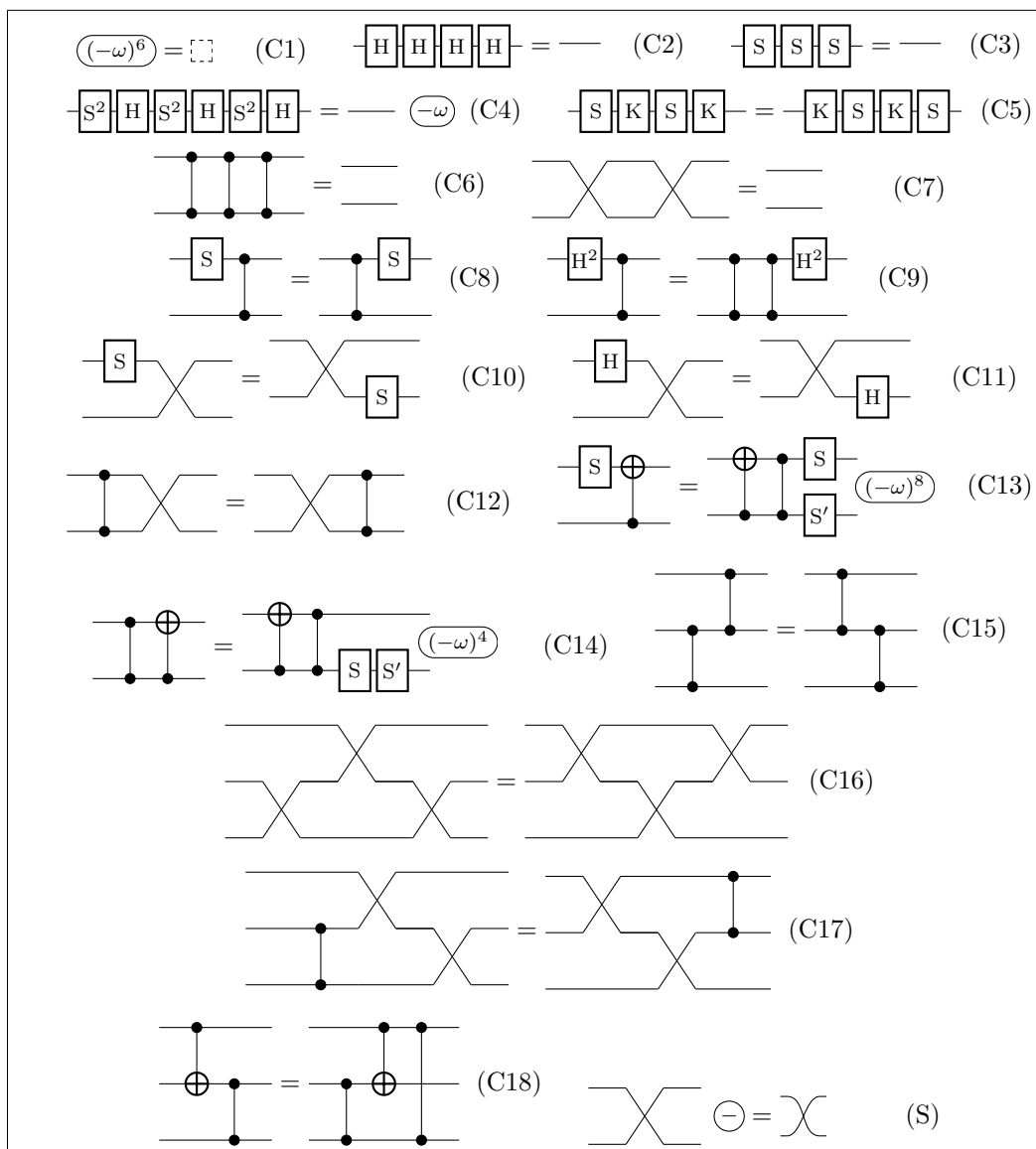


Figure 13 Set of relations $RCliff_2^{source}$. Equations (R_i) are the ones of the complete set of relations for the pro considered in [?].



■ **Figure 14** Set of relations $\text{Cliff}_3^{\text{source}}$. Equations (C_i) are the ones of the complete set of relations for the pro considered in [?].

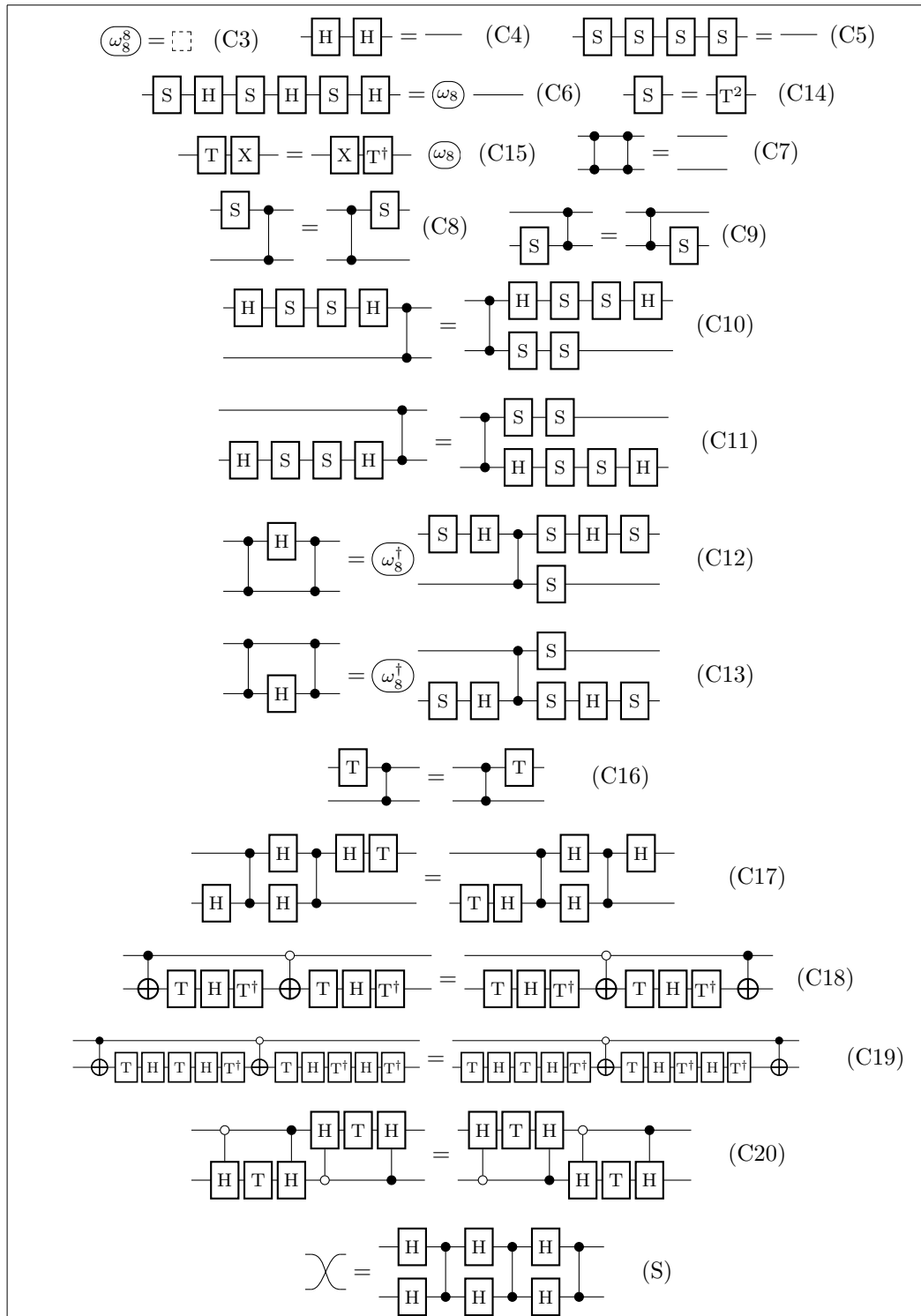


Figure 15 Set of relations $\text{Cliff}_2^{\text{source}}$ on two qubits. Equations (C_i) are the ones of the complete set of relations for the pro considered in [?].

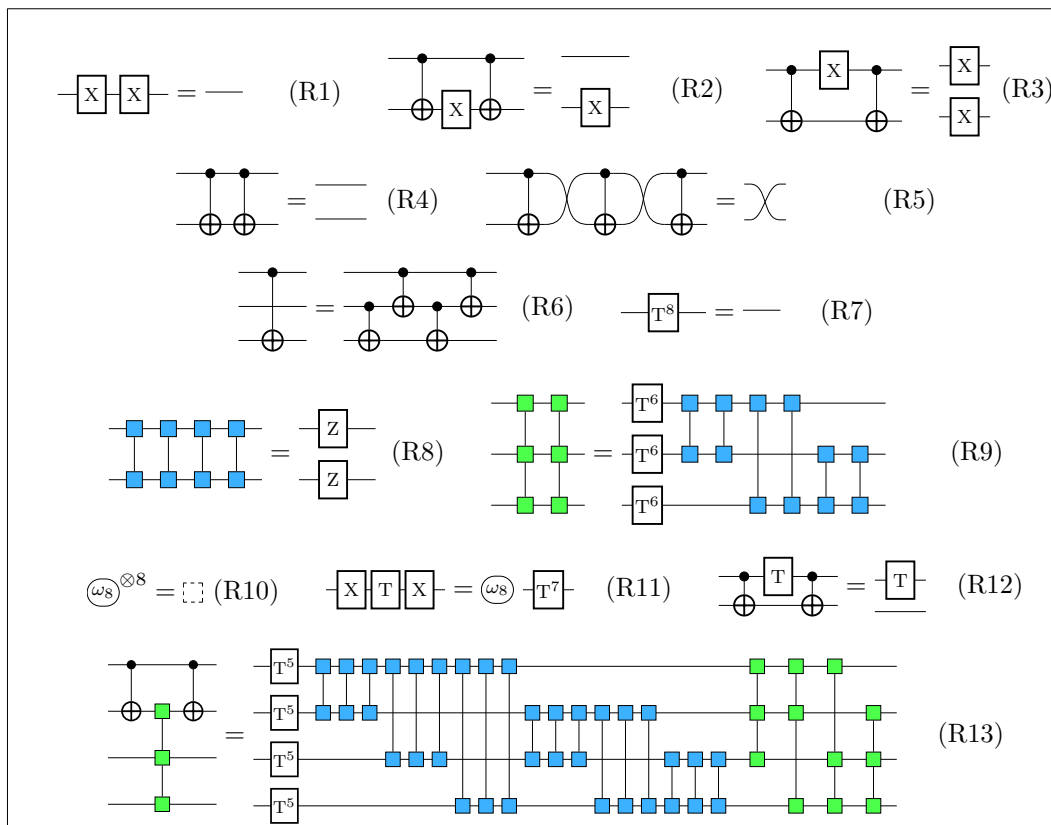


Figure 17 Set of relations $\text{CNOTdih}_2^{\text{source}}$ [?].